National
Qualifications
2023

2023 Mathematics of Mechanics

Advanced Higher

Finalised Marking Instructions

General marking principles for Advanced Higher Mathematics of Mechanics

Always apply these general principles. Use them in conjunction with the detailed marking instructions, which identify the key features required in candidates' responses.

The marking instructions for each question are generally in two sections:

generic scheme — this indicates why each mark is awarded
illustrative scheme — this covers methods which are commonly seen throughout the marking

In general, you should use the illustrative scheme. Only use the generic scheme where a candidate has used a method not covered in the illustrative scheme.

(a) Always use positive marking. This means candidates accumulate marks for the demonstration of relevant skills, knowledge and understanding; marks are not deducted for errors or omissions.

(b) If you are uncertain how to assess a specific candidate response because it is not covered by the general marking principles or the detailed marking instructions, you must seek guidance from your team leader.

(c) One mark is available for each •. There are no half marks.

(d) If a candidate's response contains an error, all working subsequent to this error must still be marked. Only award marks if the level of difficulty in their working is similar to the level of difficulty in the illustrative scheme.

(e) Only award full marks where the solution contains appropriate working. A correct answer with no working receives no mark, unless specifically mentioned in the marking instructions.

(f) Candidates may use any mathematically correct method to answer questions, except in cases where a particular method is specified or excluded.

(g) If an error is trivial, casual or insignificant, for example $6 \times 6 = 12$, candidates lose the opportunity to gain a mark, except for instances such as the second example in point (h) below.

(h) If a candidate makes a transcription error (question paper to script or within script), they lose the opportunity to gain the next process mark, for example

This is a transcription error and so the mark is not awarded.

This is no longer a solution of a quadratic equation, so the mark is not awarded.

$$x^2 + 5x + 7 = 9x + 4$$
$$x - 4x + 3 = 0$$
$$x = 1$$

The following example is an exception to the above

This error is not treated as a transcription error, as the candidate deals with the intended quadratic equation. The candidate has been given the benefit of the doubt and all marks awarded.

$$x^2 + 5x + 7 = 9x + 4$$
$$x - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 1 \text{ or } 3$$

(i) Horizontal/vertical marking

If a question results in two pairs of solutions, apply the following technique, but only if indicated in the detailed marking instructions for the question.

Example:

	$\bullet^5$	$\bullet^6$
$\bullet^5$	$x = 2$	$x = -4$
$\bullet^6$	$y = 5$	$y = -7$

Horizontal: $\bullet^5$ $x = 2$ and $x = -4$ Vertical: $\bullet^5$ $x = 2$ and $y = 5$

$\bullet^6$ $y = 5$ and $y = -7$ $\bullet^6$ $x = -4$ and $y = -7$

You must choose whichever method benefits the candidate, **not** a combination of both.

(j) In final answers, candidates should simplify numerical values as far as possible unless specifically mentioned in the detailed marking instruction. For example

$\frac{15}{12}$ must be simplified to $\frac{5}{4}$ or $1\frac{1}{4}$ $\frac{43}{1}$ must be simplified to 43

$\frac{15}{0\cdot3}$ must be simplified to 50 $\frac{4/5}{3}$ must be simplified to $\frac{4}{15}$

$\sqrt{64}$ must be simplified to 8*

*The square root of perfect squares up to and including 144 must be known.

(k) Commonly Observed Responses (COR) are shown in the marking instructions to help mark common and/or non-routine solutions. CORs may also be used as a guide when marking similar non-routine candidate responses.

(l) Do not penalise candidates for any of the following, unless specifically mentioned in the detailed marking instructions:

- working subsequent to a correct answer
- correct working in the wrong part of a question
- legitimate variations in numerical answers/algebraic expressions, for example angles in degrees rounded to nearest degree
- omission of units
- bad form (bad form only becomes bad form if subsequent working is correct), for example

  $\left(x^3+2x^2+3x+2\right)\left(2x+1\right)$ written as

  $\left(x^3+2x^2+3x+2\right)\times 2x+1$

  $=2x^4+5x^3+8x^2+7x+2$

  gains full credit

- repeated error within a question, but not between questions or papers

(m) In any 'Show that…' question, where candidates have to arrive at a required result, the last mark is not awarded as a follow-through from a previous error, unless specified in the detailed marking instructions.

(n) You must check all working carefully, even where a fundamental misunderstanding is apparent early in a candidate's response. You may still be able to award marks later in the question so you must refer continually to the marking instructions. The appearance of the correct answer does not necessarily indicate that you can award all the available marks to a candidate.

(o) You should mark legible scored-out working that has not been replaced. However, if the scored-out working has been replaced, you must only mark the replacement working.

(p) If candidates make multiple attempts using the same strategy and do not identify their final answer, mark all attempts and award the lowest mark. If candidates try different valid strategies, apply the above rule to attempts within each strategy and then award the highest mark.

For example:

Strategy 1 attempt 1 is worth 3 marks.	Strategy 2 attempt 1 is worth 1 mark.
Strategy 1 attempt 2 is worth 4 marks.	Strategy 2 attempt 2 is worth 5 marks.
From the attempts using strategy 1, the resultant mark would be 3.	From the attempts using strategy 2, the resultant mark would be 1.

In this case, award 3 marks.

Marking Instructions for each question

Question			Generic scheme	Illustrative scheme	Max mark
1.			•¹ use conservation of momentum •² calculate velocity. •³ calculate magnitude of velocity	•¹ $48\times16\mathbf{i}=48(4\mathbf{i}-8\mathbf{j})+32\mathbf{v}$ •² $18\mathbf{i}+12\mathbf{j}$ •³ $21.6\,\mathrm{cms}^{-1}$	3

Notes:
1. Candidates who convert distances to metres and give a final answer in metres per second receive full credit
2. A final answer of $21.6\,\mathrm{ms}^{-1}$ does not gain •³

Question			Generic scheme	Illustrative scheme	Max mark
2.			•¹ differentiate natural logarithm •² differentiate secant using chain rule. •³ simplify	•¹ $\frac{1}{\sec 2x}$ •² $...2\sec 2x\tan 2x$ •³ $2\tan 2x$	3

Notes:

Question			Generic scheme	Illustrative scheme	Max mark
3.	(a)		•[1] apply condition for maximum height and state initial vertical velocity •[2] substitute into equation of motion and complete	•[1] $v=0,\ u=U\sin\theta$ **stated or implied by •[2]** •[2] $0=(U\sin\theta)^2+2(-g)H$ **leading to** $H=\dfrac{U^2\sin^2\theta}{2g}$	**2**
Alternative method for (a)					
	(a)		•[1] apply condition for maximum height and rearrange to find expression for time •[2] substitute into expression for height and complete	•[1] $-gt+U\sin\theta=0$ $\Rightarrow t=\dfrac{U\sin\theta}{g}$ •[2] $H=U\left(\dfrac{U\sin\theta}{g}\right)\sin\theta-\dfrac{1}{2}g\left(\dfrac{U\sin\theta}{g}\right)^2$ $\Rightarrow H=\dfrac{U^2\sin^2\theta}{2g}$	**2**
Notes: • Treat $y=\dfrac{U^2\sin^2\theta}{2g}$ as bad form and award •[2]					
	(b)		•[3] find the maximum height from level ground •[4] interpret new situation	•[3] 16.8 m •[4] 33.2 m	**2**
Notes: 1. Candidates may not ever calculate the maximum height from level ground prior to a correct answer. If this is the case, then •[3] can be awarded for $\dfrac{40^2\sin^2 27}{2\times 9.8}$					

Question			Generic scheme	Illustrative scheme	Max mark
4.	(a)		•[1] differentiate to find velocity •[2] substitute for time	•[1] $\mathbf{2i}+\mathbf{6}t\,\mathbf{j}-\mathbf{10}t\mathbf{k}$ •[2] $\mathbf{2i}+\mathbf{18j}-\mathbf{30k}$ ms^{-1}	**2**
Notes: 1. Candidates gain no marks for substituting values into the original vector from the question or an integral of it					
	(b)		•[3] find expression for speed •[4] solve for t	•[3] $\sqrt{2^2+(6t)^2+(-10t)^2}$ **stated or implied by •[4]** •[4] $4+36t^2+100t^2=2500$ $\Rightarrow 4.28\,\text{s}$	**2**
Notes: 2. award •[4] if negative value is also stated					

Question			Generic scheme	Illustrative scheme	Max mark
5.			$\bullet^1$ differentiate u with respect to x $\bullet^2$ evaluate new limits $\bullet^3$ find new integral $\bullet^4$ evaluate	$\bullet^1$ $\frac{du}{dx} = \sec^2 x$ $\bullet^2$ $\int_0^{\sqrt{3}} \ldots du$ $\bullet^3$ $\int_0^{\sqrt{3}} u^2\, du$ $\bullet^4$ $\sqrt{3}$ or 1.73	4

Notes:

1. Award $\bullet^2$ and $\bullet^3$ if no limits appear and candidate returns to original variable at $\bullet^4$.
2. For incorrect limits at $\bullet^2$ where candidate returns to original variable at $\bullet^4$, withhold $\bullet^2$.
3. For incorrect limits at $\bullet^2$ where candidate does not return to original variable at $\bullet^4$, withhold $\bullet^2$ and $\bullet^4$.
4. Where candidates attempt to integrate an expression containing both u and x, only $\bullet^1$ and $\bullet^2$ are available.

Question			Generic scheme	Illustrative scheme	Max mark
6.	(a)		•¹ calculate ω •² calculate amplitude	•¹ 16 **stated or implied by** •² •² $\frac{1}{8}$ **or** 0.125 metres	**2**
	(b)		•³ calculate maximum acceleration	•³ 32 ms^{-2}	**1**

Notes:

1. Do not accept -32 or ±32 for •³.

Question			Generic scheme	Illustrative scheme	Max mark
7.			•¹ evidence use of quotient rule with denominator and one term of numerator correct •² complete differentiation •³ set differential equal to zero and solve	•¹ $\frac{(t^2+3)\times5-\ldots}{(t^2+3)^2}$ or $\frac{\ldots-5t\times2t}{(t^2+3)^2}$ •² $\frac{(t^2+3)\times5-5t\times2t}{(t^2+3)^2}$ •³ $\sqrt{3}$	**3**

Notes:

1. Do not accept $\pm\sqrt{3}$ for •³.

Question			Generic scheme	Illustrative scheme	Max mark
8.	(a)		•[1] determine the position of the boat at time t	•[1] $\begin{pmatrix} 4t+5 \\ t+2 \end{pmatrix}$	5
			•[2] determine the relative position of the boat and the whale	•[2] $\begin{pmatrix} 4t-55 \\ t-38 \end{pmatrix}$	
			•[3] determine the magnitude squared	•[3] $(4t-55)^2+(t-38)^2$	
			•[4] differentiate and equate to 0	•[4] $34t-516=0$ or equivalent.	
			•[5] determine the minimum distance	•[5] 23.5 or $\frac{97\sqrt{17}}{17}$ metres.	
Notes:					
	(b)		•[6] state appropriate assumption	•[6] eg the whale is a point **OR** the whale remains stationary	1
Notes:					

Question			Generic scheme	Illustrative scheme	Max mark
9.	(a)		Method 1	Method 1	**2**
			•1 recognise numerator as differential of denominator and integrate	•1 $\ln\left(2t^2+17t+8\right)+c$	
			•2 calculate value of constant	•2 $c=\ln\frac{1}{2}$ or equivalent.	
			Method 2	Method 2	
			•1 rewrite using partial fractions and integrate	•1 $\ln\left(2t+1\right)+\ln\left(t+8\right)+c$	
			•2 calculate value of constant	•2 $c=\ln\frac{1}{2}$ or equivalent.	
Notes: 1. Do not penalise the omission of $+c$ at •1.					
	(b)		•3 calculate displacement	•3 $\ln\left(\frac{77}{2}\right)$m or 3.65 m	**1**
Notes:					

Question			Generic scheme	Illustrative scheme	Max mark
10.	(a)		•[1] use appropriate equation of motion and substitute •[2] find displacement of particle D •[3] find initial speed of particle D	•[1] $0 = 10.5^2 - 2gs$ •[2] 3.625 **or** $\frac{29}{8}$ •[3] $8.43\,\text{ms}^{-1}$ **or** $\frac{7\sqrt{145}}{10}\text{ms}^{-1}$	**3**
Notes:					
	(b)		•[4] find expressions for height at any time t •[5] equate expressions and find value of t •[6] find velocity of C at this time •[7] find velocity of D at this time	•[4] $s_C = 10.5t - \frac{1}{2}gt^2$ $s_D = 2 + 8.43t - \frac{1}{2}gt^2$ •[5] $10.5t - \frac{1}{2}gt^2 = 2 + 8.43t - \frac{1}{2}gt^2$ $t = 0.966\,\text{secs}$ •[6] $1.04\,\text{ms}^{-1}$ •[7] $-1.04\,\text{ms}^{-1}$	**4**

Notes:

1. If this question is done with fractions throughout then speeds are identical in (b). However, candidates are likely to round answers in (a). Accept appropriate rounding throughout this question and in particular when stating speeds are equal. Candidates must show that bodies are moving in opposite directions.
2. •[7] can be awarded for numerical values that are equal to 2 significant figures and differ in sign.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
11.	(a)		•1 calculate displacement	•1 $\begin{pmatrix} -4 \\ 16 \end{pmatrix}$	**2**
			•2 calculate work done	•2 6.8 J	
Notes:					
	(b)		•3 calculate distance	•3 16.49 **or** $4\sqrt{17}$	**2**
			•4 calculate component of $\mathbf{F}$	•4 0.412 **or** $\frac{\sqrt{17}}{10}$ N	
Notes:					

Question			Generic scheme	Illustrative scheme	Max mark
12.			$\bullet^1$ start process of implicit differentiation	$\bullet^1$ $\ldots + 2y\dfrac{dy}{dx}\ldots - 4\dfrac{dy}{dx}\ldots$	**5**
			$\bullet^2$ complete implicit differentiation	$\bullet^2$ $3x^2 + 2y\dfrac{dy}{dx} + 2 - 4\dfrac{dy}{dx} = 0$	
			$\bullet^3$ determine expression for $\dfrac{dy}{dx}$ in terms of x and y	$\bullet^3$ $\dfrac{dy}{dx} = \dfrac{3x^2+2}{4-2y}$ or equivalent	
			$\bullet^4$ calculate values for k when x = 2	$\bullet^4$ -3, 7	
			$\bullet^5$ determine value of k	$\bullet^5$ when $k=-3$, $\dfrac{dy}{dx} > 0$,	

Notes:

1. The correct value of k must be selected for $\bullet^5$ to be awarded

Question			Generic scheme	Illustrative scheme	Max mark
13.			•[1] use Newton's inverse law to determine constant •[2] use Newton's inverse law to determine acceleration due to gravity of satellite •[3] substitute into equation for angular velocity •[4] use equation for period leading to answer	•[1] $\frac{GmM}{R^2} = 3m$ **stated or implied by •[2]** •[2] $g_s = \frac{1}{12}$ •[3] $\frac{1}{12} = \omega^2 \times 6R$ •[4] $\frac{2\pi}{\sqrt{\frac{1}{72R}}}$ leading to $12\pi\sqrt{2R}$	4

Notes:

1. Candidates may not ever calculate the acceleration due to gravity of satellite prior to a correct answer. If this is the case, then •[2] can be awarded for $\frac{3mR^2}{(6R)^2} = mg_s$

Question			Generic scheme	Illustrative scheme	Max mark
14.			•1 solve auxiliary equation	•1 $m=-\frac{2}{3}$ **stated or implied by •2**	5
			•2 state general solution	•2 $y=Ae^{-\frac{2}{3}x}+Bxe^{-\frac{2}{3}x}$	
			•3 differentiate	•3 $\frac{dy}{dx}=-\frac{2}{3}Ae^{-\frac{2}{3}x}+Be^{-\frac{2}{3}x}-\frac{2}{3}Bxe^{-\frac{2}{3}x}$ **stated or implied by •4**	
			•4 form equations and solve for a constant	•4 $A=6$ or $B=1$	
			•5 find second constant and state particular solution	•5 $y=6e^{-\frac{2}{3}x}+xe^{-\frac{2}{3}x}$	

Notes:

1. If the general solution is incorrectly stated at •2 as $y=Ae^{-\frac{2}{3}x}+Be^{-\frac{2}{3}x}$ then only •1 is available

Commonly Observed Responses:

For incorrect differentiation eg $\frac{dy}{dx}=-\frac{2}{3}Ae^{-\frac{2}{3}x}-\frac{2}{3}Bxe^{-\frac{2}{3}x}$, •4 is available as a follow through but •5 is unavailable

Question			Generic scheme	Illustrative scheme	Max mark
15.			•[1] state momentum before or after collision •[2] use conservation of linear momentum for system •[3] state initial or final energy •[4] use conservation of energy •[5] combine equations and rearrange	•[1] mu or $(M+m)v$ **stated or implied by •[2]** •[2] $mu=(M+m)v$ •[3] $\frac{1}{2}(M+m)v^2$ **or** $(M+m)gh$ **stated or implied by •[4]** •[4] $(M+m)gh=\frac{1}{2}(M+m)v^2$ •[5] $(M+m)gh=\frac{1}{2}(M+m)\left(\frac{mu}{M+m}\right)^2$ leading to $h=\frac{1}{2g}\left(\frac{mu}{M+m}\right)^2$	5

Notes:

1. Only •[1] and •[2] are available for an approach involving equations of motion rather than momentum

Question			Generic scheme	Illustrative scheme	Max mark
16.	(a)		•[1] resolve forces parallel to the slope	•[1] $T = \mu R + 3g\sin 50°$ **stated or implied by •[2]**	**3**
			•[2] resolve forces perpendicular to slope and combine equations for both objects	•[2] $3.4g = 3g\mu\cos 50° + 3g\sin 50°$	
			•[3] calculate the coefficient of friction	•[3] 0.571	

Notes:

Question			Generic scheme	Illustrative scheme	Max mark
	(b)		•[4] use Newton's second law along the line of slope	•[4] $ma = mg\sin\theta - \mu R$	**3**
			•[5] find acceleration	•[5] $g\sin 50° - \mu g\cos 50°$	
			•[6] calculate the time taken starting from rest	•[6] 2.02 seconds	

Notes:

1. Do not accept 2 or ± 2.02 for •[6]

Question			Generic scheme	Illustrative scheme	Max mark
17.	(a)		•[1] start integration by parts	•[1] $x\left(-\frac{1}{2}\cos 2x\right)\ldots$	3
			•[2] complete first application	•[2] $-\int -\frac{1}{2}\cos 2x\,dx$	
			•[3] complete integration	•[3] $-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+c$	

Notes:

1. Do not penalise the omission of dx at •[2]
2. •[3] cannot be awarded if the constant of integration is omitted

Question			Generic scheme	Illustrative scheme	Max mark
	(b)		•[4] substitute for volume of revolution	•[4] $\pi\int_0^1 x\sin 2x\,dx$	3
			•[5] substitute limits	•[5] $\pi\left[\left(-\frac{1}{2}\cos 2+\frac{1}{4}\sin 2\right)-0\right]$	
			•[6] evaluate volume	•[6] 1.37	

Notes:

3. Do not penalise the omission of dx at •[4]
4. At •[6] do not accept a multiple of π
5. If an earlier error leads to a negative volume, •[6] can only be awarded if the candidate subsequently acknowledges that the volume can't be negative

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
18.	(a)		•[1] consider energy at top of circle •[2] equate to energy at bottom of circle where max speed occurs •[3] find expression for max speed	•[1] $\frac{1}{2}m\left(2\sqrt{3gr}\right)^2+2mgr$ •[2] $\frac{1}{2}m\left(2\sqrt{3gr}\right)^2+2mgr=\frac{1}{2}mu^2$ •[3] $4\sqrt{gr}$ ms^{-1}	**3**
Notes: 1. Do not penalise $\sqrt{16gr}$ at •[3].					
	(b)	(i)	•[4] consider forces towards the centre •[5] consider energy at this point •[6] calculate height in terms of r and θ •[7] substitute for h, u and v •[8] rearrange to show $\cos\theta$	•[4] $-mg\cos\theta=\frac{mv^2}{r}$ •[5] $\frac{1}{2}mu^2=\frac{1}{2}mv^2+mgh$ •[6] $h=r-r\cos\theta$ •[7] $2mgr=-\frac{1}{2}mgr\cos\theta$ $+mgr(1-\cos\theta)$ •[8] $\ldots\Rightarrow\cos\theta=-\frac{2}{3}$	**5**
Notes: 1. •[8] can only be awarded for appropriate working prior to the statement of the answer.					
		(ii)	•[9] Statement about motion	•[9] particle now behaves as a projectile **OR** the only force acting on the particle is gravity so motion is parabolic	**1**
Notes:					

Question			Generic scheme	Illustrative scheme	Max mark
19.	(a)		•1 Use Hooke's Law with substitution •2 Resolve forces vertically and obtain expression	•1 $\frac{2mgx}{l}$ •2 $\frac{2mgx}{l} = mg$ **leading to** $\frac{l}{2}$	2
Notes: 1. An approach involving elastic potential energy gains no marks as it is not valid					
	(b)		•3 equate lengths •4 resolve forces vertically •5 combine equations and substitute •6 obtain expression	•3 $2l + x_1 + x_2 = 3l$ **stated or implied by •5** •4 $T_1 = T_2 + mg$ •5 $\frac{2mgx_1}{l} = \frac{2mg(l - x_1)}{l} + mg$ •6 $\frac{3l}{4}$	4
Notes:					
Commonly Observed Responses:					

[END OF MARKING INSTRUCTIONS]