2012 Applied Mathematics

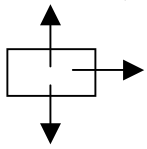
Advanced Higher – Mechanics

Finalised Marking Instructions

Advanced Higher Applied Mathematics 2012

Mechanics Solutions

A1.



Resolving: horizontally vertically

$$\sum F_x = ma \qquad \sum F_y = 0$$

$$F = \frac{mv^2}{r} \qquad N = mg$$ **1 for both equations**

$$F = \mu N$$

$$\Rightarrow \mu mg = \frac{mv^2}{r}$$ **1**

$$\mu = \frac{v^2}{gr}$$

$$= \frac{(80 \times 10^3 \div 3600)^2}{9{\cdot}8 \times 150}$$ **1**

$$= 0{\cdot}34$$

A2. Let the angle of projection be θ and the initial speed be V.

Vertically: using $v = u + at$, the maximum height occurs when $v = 0$ so $t = \dfrac{V\sin\theta}{g}$. **1**

Using $s = ut + \frac{1}{2}at^2$, the maximum height is

$$V\sin\theta\frac{V\sin\theta}{g} + \frac{1}{2}(-g)\left(\frac{V\sin\theta}{g}\right)^2 = \frac{V^2\sin^2\theta}{2g}. \qquad \textbf{1}$$

Also using $s = ut + \frac{1}{2}at^2$ and putting $s = 0$, the time of flight is $\dfrac{2V\sin\theta}{g}$. **1**

Horizontally: the speed is $V\cos\theta$ so the range is

$$\frac{2V\sin\theta}{g} \times V\cos\theta = \frac{2V^2\sin\theta\cos\theta}{g}. \qquad \textbf{1}$$

Finally:

$$\frac{V^2\sin^2\theta}{2g} = \frac{1}{10} \times \frac{2V^2\sin\theta\cos\theta}{g}$$

$$\tan\theta = \frac{2}{5}.$$

Thus the angle of projection is $\tan^{-1}\frac{2}{5}$ **1**

$$\approx 21{\cdot}8^\circ.$$

Note: Some candidates may use the formula $s = \left(\frac{u+v}{2}\right)t$.

A3. First calculate the distance he covers in $0 \leqslant t \leqslant 4$:

$$\frac{ds}{dt} = \frac{t(13 - 2t)}{2}$$

$$\Rightarrow s = \frac{1}{2}\int_0^4 \left(13t - 2t^2\right) dt \qquad \textbf{1}$$

$$s = \frac{1}{2}\left[\frac{13}{2}t^2 - \frac{2}{3}t^3\right]_0^4 = \frac{1}{2}\left[104 - \frac{128}{3}\right]$$

$$= \frac{92}{3} \text{ metres} \qquad \textbf{1}$$

During the next 6 seconds, he covers 60 metres
making a total of $90\frac{2}{3}$ metres. **1**

In the final phase, his initial speed is 10 m s^{-1}, his acceleration is $-0{\cdot}4 \text{ m s}^{-2}$ and he covers $9\frac{1}{3}$ metres. **1**

So, using $s = ut + \frac{1}{2}at^2$, we have

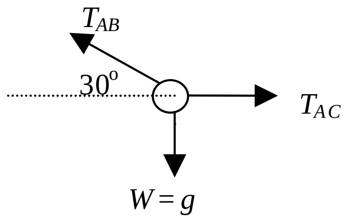
$$\frac{28}{3} = 10t - \frac{t^2}{5} \qquad \textbf{1}$$

$$\Rightarrow 3t^2 - 150t + 140 = 0$$

$$\Rightarrow t = \frac{150 \pm \sqrt{22500 - 4 \times 3 \times 140}}{2 \times 3} \approx 0{\cdot}951,\ 49{\cdot}049$$

Taking the smaller of these gives his total time as $4 + 6 + 0{\cdot}95 = 10.95$ seconds. **1**

A4.



(a) Resolving vertically:

$$\sum F_y = 0$$

$$T_{AB}\sin 30° - g = 0 \qquad \textbf{1}$$

$$T_{AB} = \frac{g}{\sin 30°} = 2g$$

Resolving horizontally:

$$\sum F_x = 0$$

$$T_{AC} - T_{AB}\cos 30° = 0 \qquad \textbf{1}$$

$$T_{AC} = T_{AB}\cos 30°$$

$$= 2g\frac{\sqrt{3}}{2}$$

$$= \sqrt{3}g \approx 16{\cdot}97 \text{ N} \qquad \textbf{1}$$

(b)

$$T_{AC} = \frac{\lambda x}{l}$$

$$x = \frac{T_{AC}l}{\lambda}$$

$$= \frac{9{\cdot}8\sqrt{3} \times 0{\cdot}10}{40} \qquad \textbf{1}$$

$$= 0{\cdot}0424$$

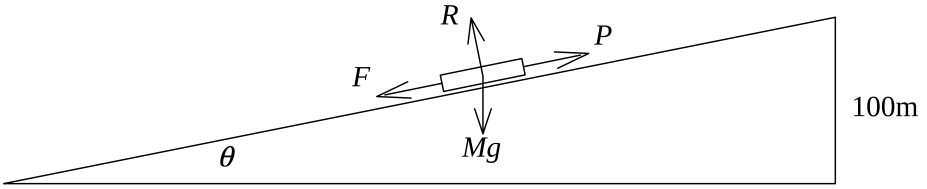
$$AC = l + x$$

$$= 0{\cdot}10 + 0{\cdot}0424 \qquad \textbf{1}$$

$$= 0{\cdot}1424$$

The distance *AC* is 0·142 m (or 14·2 cm).

A5.



Take the mass of the train to be M kg.

Solution 1

Resolving perpendicular to the slope: $R = Mg\cos\theta$ **1**

Resolving parallel to the slope:
$s = 1000$, $u = 4$, $v = 10$ so to find the acceleration:

$$v^2 = u^2 + 2as \Rightarrow 100 = 16 + 2000a$$ **1M**

$$\Rightarrow a = 0{\cdot}042$$ **1**

By Newton's second law

$$P - F - Mg\sin\theta = Ma$$ **1M**

$$120\,000 - 0{\cdot}2 \times Mg\cos\theta - Mg\sin\theta = 0{\cdot}042M$$

$$120\,000 - (0{\cdot}199 + 0{\cdot}1)Mg = 0{\cdot}042M$$ **1**

$$(0{\cdot}042 + 2{\cdot}930)M = 120\,000$$

$$M \approx 40\,400 \text{ kg}$$ **1**

Solution 2

Using the work-energy principle. **1M**

Initial energy = $\frac{1}{2}M \times 4^2 = 8M$.
Final energy = $Mg \times 100 + \frac{1}{2}M \times 10^2 = 1030M$ **1**

Normal reaction: $R = Mg\cos\theta$
Net forward force = $P - \mu R = 120\,000 - 0{\cdot}2 \times Mg\cos\theta = 120\,000 - 1{\cdot}95M$ **1**

Work done = force × distance = $(120\,000 - 1{\cdot}95M) \times 1000$ **1**

$$\text{Change in energy} = \text{Work done}$$

$$1030M - 8M = 1000(120\,000 - 1{\cdot}95M)$$ **1**

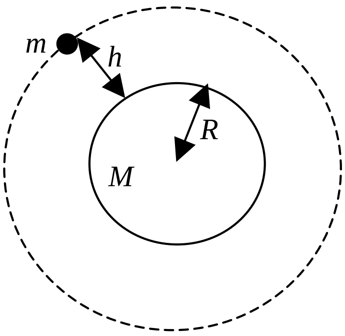
$$1022M + 1950M = 12\,000\,000$$

$$2972M = 12\,000\,000$$

$$M = \frac{12\,000\,000}{2972}$$

$$\approx 40\,400 \text{ kg}$$ **1**

A6. (a)



$$F = \frac{GMm}{r^2} = \frac{mv^2}{r} \qquad \textbf{1M}$$

$$v^2 = \frac{GM}{r}$$

Let the radius of the Earth be R km.

Then, at the surface, $\frac{GMm}{R^2} = mg \Rightarrow GM = gR^2.$ **1**

$$\therefore \quad v^2 = \frac{gR^2}{(R + h)} \qquad \textbf{1}$$

$$= \frac{9{\cdot}8 \times (6380 \times 10^3)^2}{(6380 + 390) \times 10^3}$$

$$= 5{\cdot}892 \times 10^7$$

$$\Rightarrow v \approx 7676 \text{ m s}^{-1} \qquad \textbf{1}$$

(b)

$$C = 2\pi(R + h) = 2\pi(6380 + 390) \times 10^3$$

$$= 4{\cdot}254 \times 10^7 \text{ metres} \qquad \textbf{1}$$

$$t = \frac{C}{v} = \frac{4{\cdot}254 \times 10^7}{7676} = 5{\cdot}542 \times 10^3 \text{ seconds} \qquad \textbf{1}$$

Time for 1 orbit $\approx 5{\cdot}542 \times 10^3$ seconds.

So the number of orbits is $\approx \frac{24 \times 60 \times 60}{5{\cdot}542 \times 10^3} \approx 15{\cdot}6$ **1**

(b) *Alternative*

$T = \frac{2\pi}{\omega}$ $\qquad v = r\omega$ **1**

$\omega = \frac{7676}{(6770 \times 10^3)} = 0{\cdot}00113$ $\qquad T = \frac{2\pi}{0{\cdot}00113}$ **1**

So the number of orbits is $\approx 15{\cdot}6$ **1**

A7. Let the initial height be h_i and the final height (after the rebound) be h_f.

Using conservation of energy:

$$\left.\begin{aligned} mgh_i &= \tfrac{1}{2}mu^2 \quad \Rightarrow \quad u^2 = 2gh_i \\ mgh_f &= \tfrac{1}{2}mv^2 \quad \Rightarrow \quad v^2 = 2gh_f \end{aligned}\right\} \qquad \textbf{1}$$

$$\left.\begin{aligned} h_i &= L - L\cos 45° = L\left(1 - \tfrac{1}{\sqrt{2}}\right) \\ h_f &= L - L\cos 30° = L\left(1 - \tfrac{\sqrt{3}}{2}\right) \end{aligned}\right\} \qquad \textbf{1}$$

$$\left.\begin{aligned} u^2 &= 2gL\left(1 - \tfrac{1}{\sqrt{2}}\right) \\ &= gL\left(2 - \sqrt{2}\right) \end{aligned}\right\} \qquad \textbf{1}$$

$$\text{i.e. } u = \sqrt{gL\left(2 - \sqrt{2}\right)}.$$

$$\left.\begin{aligned} v^2 &= 2gL\left(1 - \tfrac{\sqrt{3}}{2}\right) \\ &= gL\left(2 - \sqrt{3}\right) \end{aligned}\right\} \qquad \textbf{1}$$

$$\text{i.e. } v = \sqrt{gL\left(2 - \sqrt{3}\right)}.$$

By the conservation of linear momentum,

$$\left.\begin{aligned} mu &= -mv + MV \\ \text{or} \quad m\mathbf{u} &= m\mathbf{v} + M\mathbf{V} \end{aligned}\right\} \qquad \textbf{1M}$$

$$m\sqrt{gL\left(2 - \sqrt{2}\right)} = -m\sqrt{gL\left(2 - \sqrt{3}\right)} + MV \qquad \textbf{1}$$

$$\begin{aligned} MV &= m\sqrt{gL}\sqrt{2 - \sqrt{2}} + m\sqrt{gL}\sqrt{2 - \sqrt{3}} \\ &= m\sqrt{gL}\left(\sqrt{2 - \sqrt{2}} + \sqrt{2 - \sqrt{3}}\right) \\ &= m\sqrt{gL}\left(0{\cdot}7654 + 0{\cdot}5176\right) \end{aligned}$$

$$V \approx 1{\cdot}28\frac{m}{M}\sqrt{gL} \qquad \textbf{1}$$

$$k \approx 1{\cdot}28$$

A8. (a)

$$\mathbf{r}_P = \left(t^2 + 3\right)\mathbf{i} + 4t\mathbf{j}$$

$$\mathbf{v}_P = 2t\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{a}_P = 2\mathbf{i} \qquad \mathbf{1}$$

$$\mathbf{v}_Q = 2t\mathbf{i} + \mathbf{c}$$

$$t = 0, \mathbf{v} = -4\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{c} = \mathbf{j} - 4\mathbf{i}$$

$$\mathbf{v}_Q = (2t - 4)\mathbf{i} + \mathbf{j} \qquad \mathbf{1}$$

$$\mathbf{r}_Q = \left(t^2 - 4t\right)\mathbf{i} + t\mathbf{j} + \mathbf{d}$$

$$t = 0, \mathbf{r}_Q = 8\mathbf{j} \Rightarrow \mathbf{d} = 8\mathbf{j}$$

$$\mathbf{r}_Q = \left(t^2 - 4t\right)\mathbf{i} + (t + 8)\mathbf{j} \qquad \mathbf{1}$$

(b)

$$\mathbf{r}_P - \mathbf{r}_Q = \left(t^2 + 3 - t^2 + 4t\right)\mathbf{i} + \left(4t - t - 8\right)\mathbf{j}$$

$$= \left(4t + 3\right)\mathbf{i} + \left(3t - 8\right)\mathbf{j} \qquad \mathbf{1}$$

$$PQ^2 = (4t + 3)^2 + (3t - 8)^2 \qquad \mathbf{1}$$

$$\frac{d}{dt}\left(PQ^2\right) = 8(4t + 3) + 6(3t - 8) \qquad \mathbf{1}$$

$$= 50t - 24$$

$$= 0 \Rightarrow t = \frac{24}{50} \qquad \mathbf{1}$$

i.e. the particles are closest to each other after 0·48 seconds.

(c) The particles are moving at right angles to each other when

$$\mathbf{v}_P.\mathbf{v}_Q = 0 \qquad \mathbf{1M}$$

$$\mathbf{v}_P.\mathbf{v}_Q = \left(2t\mathbf{i} + 4\mathbf{j}\right).\left((2t - 4)\mathbf{i} + \mathbf{j}\right)$$

$$= 4t^2 - 8t + 4 \qquad \mathbf{1}$$

$$= 4(t - 1)^2$$

Perpendicular motion after 1 second. **1**

(b) *Alternative*:

At the closest point $\left(\mathbf{r}_P - \mathbf{r}_Q\right).\left(\mathbf{v}_P - \mathbf{v}_Q\right) = 0.$ **1M**

$$\mathbf{r}_P - \mathbf{r}_Q = \left(t^2 + 3 - t^2 + 4t\right)\mathbf{i} + \left(4t - t - 8\right)\mathbf{j}$$

$$= \left(4t + 3\right)\mathbf{i} + \left(3t - 8\right)\mathbf{j} \qquad \mathbf{1}$$

$$\mathbf{v}_P - \mathbf{v}_Q = \left(2t\mathbf{i} + 4\mathbf{j}\right) - \left((2t - 4)\mathbf{i} + \mathbf{j}\right)$$

$$= 4\mathbf{i} + 3\mathbf{j} \qquad \mathbf{1}$$

$$\left(\mathbf{r}_P - \mathbf{r}_Q\right).\left(\mathbf{v}_P - \mathbf{v}_Q\right) = 0$$

$$4(4t + 3) + 3(3t - 8) = 0 \Rightarrow 25t = 12$$

$$\Rightarrow t = 0{\cdot}48 \qquad \mathbf{1}$$

A9.

$$F_h - R = ma \text{ and } P_h = F_h v \qquad \mathbf{1}$$

Hence $$\frac{P_h}{v} - R = ma$$

$$P_h - Rv = mv\frac{dv}{dt}$$

$$1500 - (100 + 5v)v = 100v\frac{dv}{dt} \qquad \mathbf{1}$$

$$300 - 20v - v^2 = 20v\frac{dv}{dt} \qquad \mathbf{1}$$

$$\Rightarrow \frac{dv}{dt} = \frac{300 - 20v - v^2}{20v}$$

$$\frac{20v}{300 - 20v - v^2}\frac{dv}{dt} = 1$$

$$20\int \frac{v}{v^2 + 20v - 300}\,dv = \int dt \qquad \mathbf{1M}$$

$$\frac{20}{4}\int\left(\frac{-3}{v + 30} + \frac{1}{10 - v}\right)dv = \int dt$$

$$5\left[-3\ln|v + 30| - \ln|10 - v|\right] = t + c \qquad \mathbf{1}$$

When $t = 0, v = 0 \Rightarrow c = -15\ln 30 - 5\ln 10$ **1**

$$t = 15\ln 30 + 5\ln 10 - 15\ln|v + 30| - 5\ln|10 - v| \qquad \mathbf{1}$$

$$= 15\ln\left|\frac{30}{v + 30}\right| + 5\ln\left|\frac{10}{10 - v}\right|$$

When $v = 8$, $t = 15\ln\frac{30}{38} + 5\ln\frac{10}{2}$

$$= -3.55 + 8.05$$

$$= 4{\cdot}5 \qquad \mathbf{1}$$

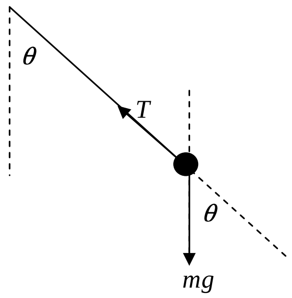
Maximum speed occurs when $\frac{dv}{dt} = 0$

i.e. $300 - 20v - v^2 = 0$

$$(30 + v)(10 - v) = 0$$

i.e. $v = 10$ **1**

A10. (a)



Component of weight perpendicular to the string is $mg\,\sin\theta$. **1**

Assuming clockwise is positive

$$-mg\,\sin\theta = mL\frac{d^2\theta}{dt^2}$$

$$-g\,\sin\theta = L\frac{d^2\theta}{dt^2} \qquad \mathbf{1}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

For small angles, $\sin\theta \approx \theta$ **1**

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta$$

Characteristic equation for SHM is $\ddot{x} = -\omega^2 x$. Since $\omega^2 = \frac{g}{L}$ **1**

$$T = \frac{2\pi}{\omega}$$

$$= 2\pi\sqrt{\frac{L}{g}} \qquad \mathbf{1}$$

When $T = 2$,

$$2 = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \frac{L}{g} = \frac{1}{\pi^2} \Rightarrow L = \frac{g}{\pi^2} \approx 0{\cdot}993 \text{ (to 3 sf)} \qquad \mathbf{1}$$

i.e. the length of the pendulum is approximately 1 metre.

(b) Let the amplitude be A metres.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ rad s}^{-1} \qquad \mathbf{1}$$

$$v^2 = \omega^2\left(A^2 - x^2\right)$$

$$16\pi^2 = \pi^2\left(A^2 - 9\right)$$

$$A^2 = 25 \qquad \mathbf{1}$$

$$A = 5$$

The amplitude is 5 metres.

$$x = A\sin\omega t$$

$$3 = 5\sin\omega t \qquad \mathbf{1}$$

$$\sin\pi t = \frac{3}{5}$$

$$t = \frac{1}{\pi}\sin^{-1}\frac{3}{5} \text{ or } 0{\cdot}20 \text{ seconds} \qquad \mathbf{1}$$

END OF SECTION A

Section B

B1. The general term is given by

$$\binom{8}{r} x^{2(8-r)}(3x)^r \qquad \mathbf{1}$$

$$= \binom{8}{r} 3^r x^{16-r} \qquad \mathbf{1,1}$$

For x^{13},

$$16 - r = 13 \Rightarrow r = 3 \qquad \mathbf{1}$$

The corresponding coefficient is

$$\frac{8!}{3!\,5!} \times 3^3 = 1512 \qquad \mathbf{1}$$

{Note: some candidates may start from: $\binom{8}{r} x^{2r}(3x)^{8-r}$ leading to $r = 5$.}

B2. (a)

$$y = \frac{x}{x^2 + 4} \Rightarrow \frac{dy}{dx} = \frac{(x^2 + 4) - x.(2x)}{(x^2 + 4)^2} \qquad \mathbf{1M, 1}$$

$$x = 2 \Rightarrow \frac{dy}{dx} = \frac{8 - 8}{8^2} = 0. \qquad \mathbf{1}$$

(b)

$$\int e^{-2t}\,dt = \left(-\frac{1}{2}\right)e^{-2t} + c \qquad \begin{cases} \mathbf{1} \text{ for } \left(-\frac{1}{2}\right) \\ \mathbf{1} \text{ for } e^{-2t} \end{cases}$$

B3. (a)

$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \mathbf{1M}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} \qquad \mathbf{1}$$

(b)

$$M^3 = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \qquad \mathbf{1}$$

$$M + M^2 + M^3 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 18 & 3 & 0 \\ 0 & 0 & \lambda + \lambda^2 + \lambda^3 \end{pmatrix} \qquad \mathbf{1}$$

(c)

$$\det M = 1 \times (1 \times \lambda) + 0 + 0 = \lambda \qquad \mathbf{1}$$

Hence the matrix M has an inverse when $\lambda \neq 0$. **1**

B4.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \qquad \textbf{1M}$$

$$1 = A(x+1) + Bx$$

$$x = 0 \quad \Rightarrow \quad A = 1 \qquad \textbf{1}$$

$$x = -1 \quad \Rightarrow \quad B = -1 \qquad \textbf{1}$$

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$V = \int \pi y^2 dx \;\Rightarrow\; V = \pi \int_1^3 \left(\frac{1}{\sqrt{x^2+x}}\right)^2 dx \qquad \textbf{1M}$$

$$= \pi \int_1^3 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \qquad \textbf{1}$$

$$= \pi \left[\ln x - \ln(x+1)\right]_1^3 \qquad \textbf{1}$$

$$= \pi \left\{\left[\ln 3 - \ln 4\right] - \left[\ln 1 - \ln 2\right]\right\}$$

$$= \pi \ln \frac{3}{2} \left(\approx 1.274 \text{ to 3 s.f.}\right) \qquad \textbf{1}$$

B5. (a)

$$\frac{dT}{dx} = k(180 - T)$$

$$\int \frac{dT}{180 - T} = \int k\, dx \qquad \textbf{1M}$$

$$-\int \frac{(-1)}{180 - T}\, dT = \int k\, dx$$

$$-\ln(180 - T) = kx + c \qquad \textbf{1}$$

Since $T = 4$ when $x = 0$

$$-\ln 176 = c \qquad \textbf{1}$$

$$\Rightarrow \ln(180 - T) - \ln 176 = -kx$$

$$\ln \frac{180 - T}{176} = -kx$$

$$\frac{180 - T}{176} = e^{-kx}$$

$$180 - T = 176e^{-kx} \qquad \textbf{1}$$

$$\text{i.e. } T = 180 - 176e^{-kx}.$$

(b) When $x = 1, T = 30$

$$e^{-k} = \frac{150}{176} \qquad \textbf{1}$$

$$\Rightarrow k \approx 0{\cdot}16 \qquad \textbf{1}$$

(c) Using $k = 0{\cdot}16$ and $T = 80$ in $T = 180 - 176e^{-kx}$ gives

$$80 = 180 - 176e^{-0{\cdot}16x} \qquad \textbf{1}$$

Hence

$$e^{-0{\cdot}16x} = \frac{100}{176}$$

$$\Rightarrow -0{\cdot}16x = \ln \frac{100}{176} \qquad \textbf{1}$$

$$\Rightarrow x \approx 3.533 \text{ hours} \approx 212 \text{ minutes} \qquad \textbf{1}$$

So the turkey should be cooked after 3 hours 32 minutes (or 212 minutes).

END OF MARKING INSTRUCTIONS