AH National Qualifications 2023 MODIFIED

X802/77/11

Mathematics of Mechanics

TUESDAY, 9 MAY

1:00 PM – 3:50 PM

Total marks — 90

Attempt ALL questions.

You may use a calculator.

To earn full marks you must show your working in your answers.

State the units for your answer where appropriate. Any rounded answer should be accurate to an appropriate number of significant figures unless otherwise stated.

Write your answers clearly in the spaces provided in the answer booklet. The size of the space provided for an answer is not an indication of how much to write. You do not need to use all the space.

Additional space for answers is provided at the end of the answer booklet. If you use this space you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not you may lose all the marks for this paper.

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FORMULAE LIST

Newton's inverse square law of gravitation

$$F = \frac{GMm}{r^2}$$

Simple harmonic motion

$$v^2 = \omega^2(a^2 - x^2)$$

$$x = a\sin(\omega t + \alpha)$$

Centre of mass

Triangle: $\frac{2}{3}$ along median from vertex.

Semicircle: $\frac{4r}{3\pi}$ along the axis of symmetry from the diameter.

Coordinates of the centre of mass of a uniform lamina, area A square units, bounded by the equation $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\overline{x} = \frac{1}{A}\int_a^b xy\,dx \qquad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}y^2\,dx$$

Standard derivatives	
$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax) + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$

MARKS

Total marks — 90

Attempt ALL questions

Note that g m s^{-2} denotes the magnitude of the acceleration due to gravity. Where appropriate, take its magnitude to be 9.8 m s^{-2}.

1. An air hockey pusher of mass 48 grams is moving freely with a velocity of $16\mathbf{i}$ cm s^{-1} when it collides with a stationary puck of mass 32 grams.

Immediately after the collision the pusher has a velocity of $(4\mathbf{i} - 8\mathbf{j})$ cm s^{-1}.

Calculate the magnitude of the velocity of the puck immediately after the collision. **3**

2. Differentiate $f(x) = \ln(\sec 2x)$ and simplify your answer. **3**

3. A projectile is launched from a point on horizontal ground with speed U m s^{-1} at an angle θ to the horizontal.

(a) Show that the maximum height, H metres, reached by the particle is given by

$$H = \frac{U^2 \sin^2 \theta}{2g}$$ **2**

(b) A particle is launched from a point h metres above horizontal ground with speed 40 m s^{-1} at an angle of 27° to the horizontal.

Calculate the value of h if the particle reaches a maximum height of 50 metres. **2**

[Turn over

MARKS

4. A particle has displacement in metres given by $2t\,\mathbf{i}+3t^2\,\mathbf{j}-5t^2\,\mathbf{k}$ where t is time in seconds and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the x, y and z axes respectively.

(a) Calculate the velocity of the particle after 3 seconds. **2**

(b) Calculate the time when the speed of the particle is 50 m s^{-1}. **2**

5. Use the substitution $u=\tan x$ to evaluate $\int_0^{\frac{\pi}{3}} \tan^2 x\sec^2 x\,dx$. **4**

6. A vertical spring has one end fixed and the other end attached to a particle.

The particle is pulled vertically downwards a small distance and released.

The ensuing motion is simple harmonic with period $\dfrac{\pi}{8}$ seconds.

As the particle passes through the equilibrium position it has a speed of 2 m s^{-1}.

(a) Calculate the amplitude of the motion. **2**

(b) Calculate the maximum acceleration of the particle. **1**

7. Given $f(t)=\dfrac{5t}{t^2+3}$, find the value of k when $f'(k)=0$, where $k>0$. **3**

MARKS

8. A boat is initially at position $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ metres.

It is moving with a constant velocity of $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ $\mathrm{m\,s^{-1}}$.

A whale is resting at the surface of the water at position $\begin{pmatrix} 60 \\ 40 \end{pmatrix}$ metres.

(a) Determine the closest distance that the boat gets to the whale. **5**

(b) State one assumption that you have made. **1**

9. (a) Find $f(t)$ if $f'(t) = \dfrac{4t+17}{2t^2+17t+8}$ and $f(0) = \ln 4$, where $t \geq 0$. **2**

(b) If $f'(t)$ represents the velocity of a particle in $\mathrm{m\,s^{-1}}$, where t is the time in seconds, calculate its displacement after 3 seconds. **1**

10. A particle C is fired vertically upwards, from horizontal ground, with speed 10.5 $\mathrm{m\,s^{-1}}$.

At the same instant another particle D is fired vertically upwards, from a point 2 metres above the ground.

(a) Find the initial speed of D given that both particles reach the same maximum height. **3**

(b) Show that, at the instant when C and D are at the same height, the particles have the same speed and are moving in opposite directions. **4**

11. A force $\mathbf{F} = 0.3\mathbf{i} + 0.5\mathbf{j}$ moves a body between two points on the $x\,y$ plane with coordinates $(16, 4)$ and $(12, 20)$.

The force is given in newtons, and distances are given in metres.

(a) Calculate the work done by the force $\mathbf{F}$. **2**

(b) Hence find the component of the force in the direction of the displacement. **2**

[Turn over

MARKS

12. A bead travels along a wire modelled by part of the curve with equation $x^3 + y^2 + 2x - 4y = 33$.

The bead passes through two points with coordinates of the form $(2, k)$.

Determine the value of k for which the gradient is positive. **5**

13. A planet has a radius of R metres. A satellite, at a height of $5R$ metres above the surface, moves in a circular orbit about the planet.

The acceleration due to gravity at the surface of the planet is $3\ \mathrm{m\,s^{-2}}$.

Show that the period of the orbit of the satellite is given by $12\pi\sqrt{2R}$ seconds. **4**

14. Solve the differential equation

$$9\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = 0$$

given that when $x = 0,\ y = 6$ and $\dfrac{dy}{dx} = -3$. **5**

15. A bullet of mass m kg is fired at a block of wood of mass M kg which hangs vertically and at rest at the end of a light inextensible string.

The bullet enters the block horizontally while travelling at a speed of $u\ \mathrm{m\,s^{-1}}$, and becomes embedded in the block.

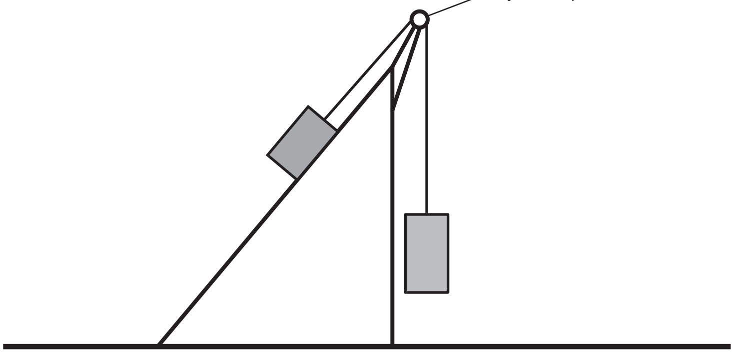
The block then swings until it reaches a height h metres above its original position.

Show that $h = \dfrac{1}{2g}\left(\dfrac{mu}{M+m}\right)^2$. **5**

MARKS

16. A box of mass 3 kg sits on a rough slope inclined at an angle of 50° to the horizontal.

The box is attached to a light inextensible string which passes up the slope and over a smooth fixed pulley to a free hanging mass of 3.4 kg.



The box is on the point of moving up the slope.

(a) Calculate the coefficient of friction between the box and the slope. **3**

The 3.4 kg mass is removed, and the box slides down the slope.

(b) Calculate the time it will take for the box to travel 8 metres down the slope, given that it does not reach the bottom of the slope during this time. **3**

17. (a) Use integration by parts to find $\int x \sin 2x \, dx$. **3**

A solid is formed by rotating the curve with equation $y = \sqrt{x \sin 2x}$ between $x = 0$ and $x = 1$ through 2π radians about the x-axis.

(b) Calculate the volume of this solid. **3**

[Turn over

MARKS

18. A particle of mass m kg is attached to a light inextensible string. The mass is rotating in a vertical circle of radius r metres. Its velocity at the top of the circle is $2\sqrt{3gr}$ m s^{-1}.

(a) Find an expression in terms of g and r for the maximum velocity of the particle. **3**

On another occasion, the same string and mass are set in motion so that the mass has velocity $2\sqrt{gr}$ m s^{-1} at the lowest point of the circle. The angle between the string and the downward vertical is θ.

(b) (i) Show that at the point where the tension in the string becomes zero, $\cos\theta = -\frac{2}{3}$. **5**

(ii) Describe the motion of the particle after this point. **1**

19. A particle of mass, m kg, is suspended in equilibrium from a point A on the ceiling of a room by a light spring of natural length l metres and modulus of elasticity $2\,mg$ newtons.

(a) Show that the extension of the spring is $\frac{l}{2}$ metres. **2**

A second identical spring is attached to the particle and secured to a point B on the floor of the room. B is vertically below A and the distance AB is $3l$ metres.

(b) Given that both springs remain in tension when the particle is again in equilibrium, find an expression in terms of l for the extension of the original spring. **4**

[END OF QUESTION PAPER]

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