AH National Qualifications 2017

X702/77/11

Mathematics of Mechanics

MONDAY, 29 MAY

1:00 PM – 4:00 PM

Total marks — 100

Attempt ALL questions.

You may use a calculator.

Full credit will be given only to solutions which contain appropriate working.

State the units for your answer where appropriate. Any rounded answer should be accurate to three significant figures (or one decimal place for angles in degrees) unless otherwise stated.

Write your answers clearly in the answer booklet provided. In the answer booklet, you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

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A/HTP

FORMULAE LIST

Newton's inverse square law of gravitation

$$F = \frac{GMm}{r^2}$$

Simple harmonic motion

$$v^2 = \omega^2(a^2 - x^2)$$

$$x = a\sin(\omega t + \alpha)$$

Centre of mass

Triangle: $\frac{2}{3}$ along median from vertex.

Semicircle: $\frac{4r}{3\pi}$ along the axis of symmetry from the diameter.

Standard derivatives	
$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax) + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$

Total marks — 100 **MARKS**

Attempt ALL questions

Candidates should observe that $g\,\mathrm{m\,s^{-2}}$ denotes the magnitude of the acceleration due to gravity. Where appropriate, take its magnitude to be $9{\cdot}8\,\mathrm{m\,s^{-2}}$.

1. A skier starts from rest and skis straight down a slope inclined at an angle θ to the horizontal, where $\sin\theta = \frac{1}{4}$. The coefficient of friction between the skis and the snow is 0·125.

Find the speed of the skier after she has travelled 75 metres. **4**

2. (a) If $f(x) = \frac{\ln x}{2x^2}$, $x \neq 0$, find $f'(x)$. Fully simplify your answer. **3**

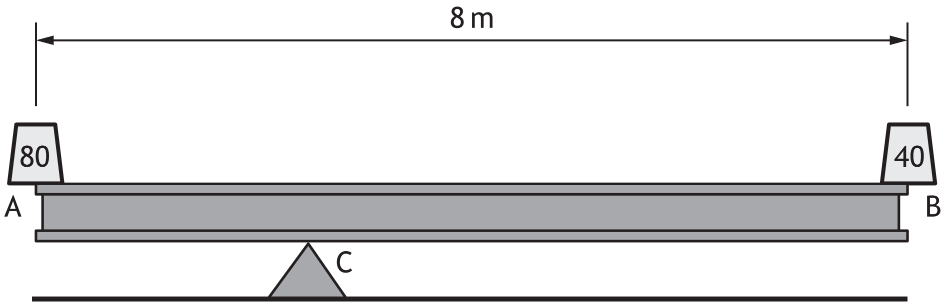
(b) If $y = \operatorname{cosec}^2 3x$, show that

$$\frac{dy}{dx} + 6y\cot 3x = 0.$$ **3**

3. The velocity of a particle after t seconds of travel can be expressed as $\mathbf{v} = (3\sin 2t)\mathbf{i} + (\cos 2t - 3)\mathbf{j}\,\mathrm{ms^{-1}}$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in horizontal and vertical directions respectively.

Find the magnitude of the acceleration of the particle when $t = \frac{\pi}{6}$ seconds. **4**

4. A uniform beam of length 8 metres has mass 200 kg and has a support placed at C. To enable it to rest horizontally, masses of 80 kg and 40 kg are attached at ends A and B as shown in the diagram.



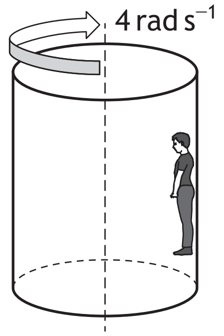
Determine the position of the support relative to the point A. **4**

[Turn over

MARKS

5. Express $\dfrac{3x^2+4x+17}{(x-3)(x^2+5)}$ as a sum of partial fractions. **4**

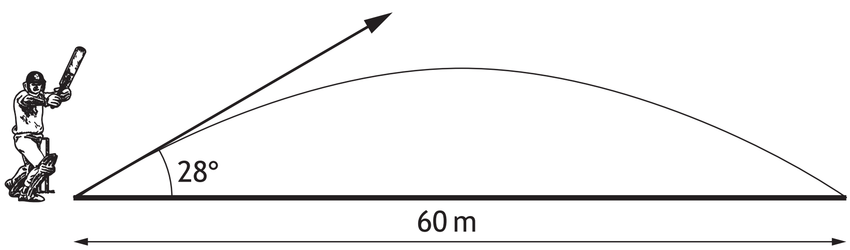
6. A ride at an amusement park consists of a hollow cylinder of radius 3·5 metres which rotates about its vertical axis of symmetry.



When the angular speed reaches 4 rad s^{-1} the floor is lowered and a person remains in contact with the inner surface of the cylinder without slipping.

What is the minimum coefficient of friction to prevent the person from slipping? **4**

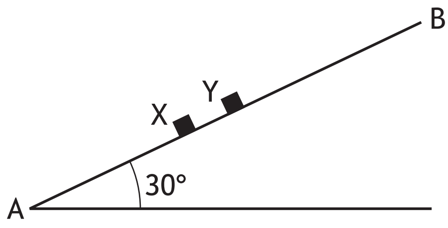
7. A cricket batsman hits a ball from ground level. The ball lands on the boundary which is 60 metres away.



If the angle of flight to the horizontal ground is 28° at the instant the ball leaves the bat, calculate the initial speed of the ball. **5**

8. Two particles, X and Y, have masses of 0·2 kg and 0·5 kg respectively.

They are moving up a smooth plane AB, inclined at 30° to the horizontal as shown in the diagram.



The particles collide 3·5 metres from B when X is moving with a speed of 6 m s^{-1} and Y is moving with a speed of 3 m s^{-1}.

This collision causes X to come instantaneously to rest while Y continues to travel up the slope.

Show that in the subsequent motion, Y comes to rest before reaching B. **6**

9. A body of mass 20 kg is moving along a rough horizontal surface with speed 12 m s^{-1}. As it passes through a point P, a horizontal force $F = \left(249 - 50\sqrt{x}\right)$ newtons is applied, where x metres is the displacement of the body from P.

Given that the coefficient of friction between the body and the surface is 0·25:

(a) find the work done on the body in the first 10 metres of its motion from P **4**

(b) find the speed of the body after travelling 10 metres from P. **2**

10. Use integration by parts to obtain $\int x^2 \sin 5x \, dx$. **5**

11. A curve is defined by $3y^2 - x^2y = 4, \quad x \geq 0,\ y \geq \dfrac{2}{\sqrt{3}}$.

Use implicit differentiation to find the gradient of the tangent when $x = 2$. **5**

[Turn over

MARKS

12. A body of mass 750 grams is attached to a light elastic string of natural length 50 cm and modulus of elasticity 150 N. The mass hangs vertically with one end of the string attached to the ceiling.

(a) Find the extension in the string when the body hangs in equilibrium. **2**

The body is released from a position 2 cm below the equilibrium position.

(b) (i) Show that the body moves with simple harmonic motion modelled by $\ddot{x} = -400x$ where x metres is the displacement from the equilibrium position. **3**

(ii) Find the speed of the body when it is 0·5 cm above the point of release. **2**

(c) On another occasion the body is pulled down 3 cm below the equilibrium position. Explain why, in this case, the subsequent motion is not simple harmonic. **1**

13. A satellite orbits the Earth at a height of h metres above its surface.

(a) If the radius of the Earth is R metres and the acceleration due to gravity experienced at the surface of the Earth is 9 times that experienced at the satellite, find an expression for h in terms of R. **4**

(b) If a second satellite is orbiting Earth at a height $3R$ metres above the surface, show that the angular velocity of the second satellite can be expressed as $\frac{1}{8}\sqrt{\frac{g}{R}}$. **3**

14. A fishing boat, A, leaves a harbour with a constant speed of 10 km h^{-1} on a bearing of 060°.

At the same time another fishing boat, B, is 12 km due east of A, moving with a constant speed of $10\sqrt{3}$ km h^{-1} on a bearing of 330°.

(a) (i) Describe how the vectors $\mathbf{i}$ and $\mathbf{j}$ should be defined in this situation. **2**

(ii) Show that the position of boat A relative to boat B, t hours after A has left the harbour, can be written as ${}_{\mathrm{A}}\mathbf{r}_{\mathrm{B}} = \left(10\sqrt{3}t - 12\right)\mathbf{i} - 10t\,\mathbf{j}$ kilometres. **3**

(b) Find for how long the two boats will be within 7 km of each other. Give your answer to the nearest minute. **5**

MARKS

15. A car of mass m kg is travelling along a straight horizontal road. It experiences resistances of total magnitude $\frac{mkv^2}{6}$, where v m s^{-1} is its velocity at any time and k is a positive constant. The engine of the car works at a constant rate P watts.

(a) Show that $\frac{dv}{dx} = \frac{6P - mkv^3}{6mv^2}$ where x metres is the displacement of the car from a fixed point O. **2**

(b) If the car starts from rest, find an expression, in terms of k, P, m and v, for the displacement of the car while it is accelerating. **4**

16. A body has a velocity v m s^{-1} and its motion after t seconds can be modelled as

$$\frac{dv}{dt} - \frac{v}{t} = 3$$

Find an expression for its velocity in terms of t, given that the body has a velocity of 5 m s^{-1} after 1 second. **5**

17. A body of mass 12 kg is moving down a rough plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{4}$. As it passes through a point A it has a speed of 5 m s^{-1}.

(a) At a point B further down the slope its speed is 10 m s^{-1}.

Show that the distance AB is $\frac{150}{\left(3 - \sqrt{7}\mu\right)g}$ metres, where μ is the coefficient of friction between the body and the plane. **5**

On reaching a speed of 10 m s^{-1} a horizontal force of 260 N is applied to the body. This brings the body to rest in a distance half that of distance AB.

(b) Calculate the value of the coefficient of friction.

Give your answer to **two significant figures**. **6**

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