2010 Applied Maths

Advanced Higher – Mechanics

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Applied Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used M and E. The code M indicates a method mark, so in question B1(a), 1M means a method mark for the product rule. The code E refers to 'error'. so in question B1(b), up to 2 marks can be awarded but 1 mark is lost for each error.

Advanced Higher Applied Mathematics 2010

Mechanics Section

A1. The displacements of the lorry and car are given by

$$s_L = Ut \qquad s_C = \frac{1}{2}at^2, \qquad \mathbf{1}$$

respectively.
The lorry and car draw level when

$$\tfrac{1}{2}at^2 = Ut \qquad \mathbf{1}$$

$$\Rightarrow t = \frac{2U}{a} \text{ (since } t > 0\text{)}. \qquad \mathbf{1}$$

The distance travelled by the car before the vehicles draw level is

$$s = U \times \frac{2U}{a} = \frac{2U^2}{a}. \qquad \mathbf{1}$$

{*Alternative methods are possible.*}

A2. Let T be the tension in the string and r the radius of the horizontal circle.
By force balance

$$T\cos\theta = 2g \qquad (*) \qquad \mathbf{1}$$

and

$$T\sin\theta = 2r\omega^2 \qquad \mathbf{1}$$

$$= 50r, \qquad \text{(since } \omega = 5\text{)}. \qquad \mathbf{1}$$

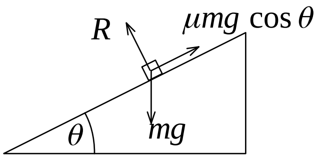
Also $r = 2\sin\theta$ so

$$T = 100 \text{ N}. \qquad \mathbf{1}$$

From (*)

$$\cos\theta = \frac{2g}{100} = \frac{g}{50} \Rightarrow \theta \approx 79°. \qquad \mathbf{1}$$

A3.



$$R = mg\cos\theta \Rightarrow \text{Friction} = \mu mg\cos\theta \qquad \mathbf{1}$$

The unbalanced force down the plane is

$$F = mg\sin\theta - \mu mg\cos\theta. \qquad \mathbf{1}$$

The acceleration of the sledge is then

$$a = g(\sin\theta - \mu\cos\theta). \qquad \mathbf{1}$$

Using $v^2 = u^2 + 2as$ gives

$$V^2 = 2g(\sin\theta - \mu\cos\theta)s \qquad \mathbf{1}$$

and hence

$$s = \frac{V^2}{2g(\sin\theta - \mu\cos\theta)}.$$

A4. (a) From Newton II:

$$F = ma$$

$$\Rightarrow -8t = 100\frac{dv}{dt} \qquad \mathbf{1}$$

$$\frac{dv}{dt} = -\frac{2}{25}t.$$

Integrating

$$v = \int -\frac{2}{25}t\,dt$$

$$= -\frac{t^2}{25} + c$$

Since $v = 4$ when $t = 0$, $c = 4$ and we get

$$v = \frac{1}{25}(100 - t^2). \qquad \mathbf{1}$$

Hence the cycle comes to rest after 10 seconds. **1**

(b) Integrating

$$v = \frac{dx}{dt} = \frac{1}{25}(100 - t^2)$$

with $x = 0$ when $t = 0$ gives

$$x = \frac{1}{75}(300t - t^3). \qquad \mathbf{1}$$

Putting $t = 10$, the stopping distance is

$$\frac{1}{75}(300 \times 10 - 10^3) = \frac{80}{3}\text{ m.} \qquad \mathbf{1}$$

A5. The change in momentum is

$$m(v - u) = \int_0^1 F(t)\,dt \qquad \mathbf{1}$$

$$= 100\int_0^1 \cos\frac{\pi t}{2}dt$$

$$= \left[\frac{200}{\pi}\sin\frac{\pi t}{2}\right]_0^1 \qquad \mathbf{1}$$

$$= \frac{200}{\pi}\ \text{kg ms}^{-1}. \qquad \mathbf{1}$$

A6. (a) Given that x is the displacement of the car from its equilibrium position, the tension in the spring is

$$T = \frac{\lambda x}{l}$$

$$= 4x. \qquad \mathbf{1}$$

The acceleration of the car is given by

$$0{\cdot}25\frac{d^2x}{dt^2} = -T$$

$$= -4x \qquad \mathbf{1}$$

giving

$$\frac{d^2x}{dt^2} = -16x$$

$$= -4^2x$$

$$\text{i.e. } \omega = 4. \qquad \mathbf{1}$$

(b) The amplitude of the motion is 0·2 m. **1**

The maximum speed is

$$v_{\max} = a\omega = 0{\cdot}2 \times 4 = 0{\cdot}8\ \text{ms}^{-1}. \qquad \mathbf{1}$$

A7. By Newton II

$$ma = -mg - 25mv^2, \qquad \mathbf{1}$$

and hence

$$v\frac{dv}{ds} = -g - 25v^2. \qquad \mathbf{1}$$

Separating the variables and integrating gives

$$-s = \int\frac{v\,dv}{g + 25v^2}, \qquad \mathbf{1}$$

and so

$$-s = \tfrac{1}{50}\ln\left|g + 25v^2\right| + C. \qquad \mathbf{1}$$

Since $v = U$ when $s = 0$,

$$C = -\tfrac{1}{50}\ln\left(g + 25U^2\right). \qquad \mathbf{1}$$

The maximum height, H, is attained when $v = 0$, giving

$$H = \frac{1}{50}\ln\left(g + 25U^2\right) - \frac{1}{50}\ln g \qquad \mathbf{1}$$

$$= \frac{1}{50}\ln\left(\frac{g + 25U^2}{g}\right).$$

A8. Resolving perpendicular to the plane

$$R = mg\cos 45°$$

Resolving parallel to the plane

$$F = mg\sin 45° + \mu R$$

$$\Rightarrow F = 2g\sin 45° + \mu\left(2g\cos 45°\right) \qquad \mathbf{1}$$

The total force acting down the plane is

$$F = \sqrt{2}g + \sqrt{2}\mu g \qquad \mathbf{1}$$

$$= \sqrt{2}\left(1 + \mu\right)g.$$

The work done by the spring is then

$$W = Fs = \sqrt{2}\,(1 + \mu) \times \frac{1}{4} \qquad \mathbf{1}$$

$$= \frac{g}{2\sqrt{2}}\,(1 + \mu)\,.$$

The elastic potential energy of a spring is $E = \dfrac{\lambda x^2}{2l}$.

So the change in the elastic potential energy is given by

$$\frac{\lambda}{2} - \frac{\lambda}{2}\left(\frac{3}{4}\right)^2 \qquad \mathbf{1}$$

$$= \frac{\lambda\left(4^2 - 3^2\right)}{2 \times 4^2} = \frac{7\lambda}{32}. \qquad \mathbf{1}$$

By the work energy principle

$$\frac{7\lambda}{32} = \frac{g}{2\sqrt{2}}\,(1 + \mu)\,, \qquad \mathbf{1}$$

and hence

$$\lambda = \frac{16g}{7\sqrt{2}}\,(1 + \mu) = \frac{8\sqrt{2}g}{7}\,(1 + \mu)\,.$$

Alternative method using differential equations:

By Newton II

$$2a = \lambda x - 2g\sin 45° - \mu\left(2g\cos 45°\right) \qquad \mathbf{2E1}$$

where x is the extension of the spring.

Rewriting gives

$$2v\frac{dv}{dx} = \lambda x - \sqrt{2}\left(1 + \mu\right)g. \qquad \mathbf{1}$$

Integrating with respect to x gives

$$v^2 = \frac{\lambda x^2}{2} - \sqrt{2}\left(1 + \mu\right)gx + C. \qquad \mathbf{1}$$

Since $v = 0$ when $x = 1$, $C = -\tfrac{1}{2}\lambda + \sqrt{2}\left(1 + \mu\right)g.$ **1**

When $x = \tfrac{3}{4}$, $v = 0$, so

$$\frac{\lambda}{2} - \frac{9\lambda}{32} = \frac{\sqrt{2}}{4}\left(1 + \mu\right)g \qquad \mathbf{1}$$

and hence

$$\lambda = \frac{8\sqrt{2}g}{7}\,(1 + \mu)\,.$$

A9. (a) The equations of motion are

$$x = V\cos\theta t, \qquad y = V\sin\theta t - \tfrac{1}{2}gt^2. \qquad \mathbf{1}$$

Hence $t = \dfrac{x}{V\cos\theta}$ and eliminating t

$$y = \left(V\sin\theta \times \frac{x}{V\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2. \qquad \mathbf{1}$$

This simplifies to

$$y = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta} \qquad \mathbf{1}$$

i.e. $$y = x\tan\theta - \frac{gx^2}{2V^2}\left(1 + \tan^2\theta\right).$$

(b) Setting $x = 3h$, $y = h$ and $V = 3\sqrt{\tfrac{1}{2}gh}$ gives

$$h = 3h\tan\theta - \frac{g(3h)^2}{9gh}\left(1 + \tan^2\theta\right). \qquad \mathbf{1}$$

Hence

$$\tan^2\theta - 3\tan\theta + 2 = 0 \qquad \mathbf{1}$$

$$(\tan\theta - 2)(\tan\theta - 1) = 0 \qquad \mathbf{1}$$

$$\tan\theta = 2 \text{ or } \tan\theta = 1. \qquad \mathbf{1}$$

Knowing that $V = 3\sqrt{\frac{gh}{2}}$, from (a) when $\tan\theta = 2$

$$y = 2x - \frac{gx^2}{9gh}\left(1 + 2^2\right)$$

$$y = 2x - \frac{5x^2}{9h}. \qquad \mathbf{1}$$

Putting $y = 0$, the range, R, is given by

$$0 = 2R - \frac{5R^2}{9h}. \qquad \mathbf{1}$$

Solving gives

$$R = \frac{18h}{5}. \qquad \mathbf{1}$$

A10. (a) The velocity vector for aircraft A is

$$\mathbf{v}_A = \frac{V}{\sqrt{2}}(\mathbf{i} + \mathbf{k}) \qquad \mathbf{1}$$

and its position vector is

$$\mathbf{r}_A = \frac{Vt}{\sqrt{2}}(\mathbf{i} + \mathbf{k}). \qquad \mathbf{1}$$

(b) The position vector for aircraft B is

$$\mathbf{r}_B = \frac{Vt}{\sqrt{2}}\mathbf{j} + \mathbf{c}. \qquad \mathbf{1}$$

And using the initial position

$$\mathbf{r}_B = \frac{1}{\sqrt{2}}\left(\sqrt{2}L\mathbf{i} + (Vt - \sqrt{2}L)\mathbf{j} + 4\sqrt{2}L\mathbf{k}\right). \qquad \mathbf{1}$$

(c) (i) The distance D between the aircraft is given by

$$D^2 = |\mathbf{r}_B - \mathbf{r}_A|^2 \qquad \mathbf{1}$$

$$= \left(L - \frac{Vt}{\sqrt{2}}\right)^2 + \left(\frac{Vt}{\sqrt{2}} - L\right)^2 + \left(4L - \frac{Vt}{\sqrt{2}}\right)^2. \qquad \mathbf{1}$$

Expanding the brackets and collecting terms gives **1**

$$D^2 = \tfrac{3}{2}V^2t^2 - 6\sqrt{2}VLt + 18L^2.$$

(ii) To minimise D^2:

$$\frac{dD^2}{dt} = 3V^2t - 6\sqrt{2}\,VL. \qquad \mathbf{1}$$

Setting this derivative equal to zero gives

$$t = \frac{2\sqrt{2}L}{V} \qquad \mathbf{1}$$

and hence the minimum value of D is $\sqrt{6}L$. **1**

A11. (a) By conservation of energy

$$\frac{1}{2}mU^2 = mgR\left(1 - \cos\frac{\pi}{4}\right). \qquad \mathbf{1}$$

This becomes

$$U^2 = 2gR\left(1 - \frac{1}{\sqrt{2}}\right) \qquad \mathbf{1}$$

i.e. $$U^2 = gR\left(2 - \sqrt{2}\right).$$

(b) Again by conservation of energy

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + mgR\left(1 - \cos\theta\right), \qquad \mathbf{1}$$

so

$$v^2 = u^2 - 2gR\left(1 - \cos\theta\right) \qquad \mathbf{1}$$

i.e. $$v = \sqrt{u^2 - 2gR(1 - \cos\theta)}$$

(c) In general

$$T(\theta) = \frac{Mv^2}{R} + Mg\cos\theta \qquad \mathbf{1}$$

$$= \frac{M}{R}\left[u^2 - 2gR\left(1 - \cos\theta\right)\right] + Mg\cos\theta \qquad \text{(using (b))} \qquad \mathbf{1}$$

$$= \frac{M}{R}\left[u^2 - 2gR\right] + 3Mg\cos\theta$$

When $\theta = \frac{1}{4}\pi$

$$T\left(\frac{\pi}{4}\right) = \frac{M}{R}\left[u^2 - 2gR\right] + 3Mg\cos\frac{\pi}{4}$$

$$= \frac{M}{R}\left[u^2 - 2gR\right] + \frac{3Mg}{\sqrt{2}} \qquad \mathbf{1}$$

When $\theta = \frac{3}{4}\pi$

$$T\left(\frac{3\pi}{4}\right) = \frac{M}{R}\left[u^2 - 2gR\right] + 3Mg\cos\frac{3\pi}{4}$$

$$= \frac{M}{R}\left[u^2 - 2gR\right] - \frac{3Mg}{\sqrt{2}} \qquad \mathbf{1}$$

Hence

$$T\left(\frac{\pi}{4}\right) - T\left(\frac{3\pi}{4}\right) = \left\{\frac{M}{R}\left[u^2 - 2gR\right] + \frac{3Mg}{\sqrt{2}}\right\} - \left\{\frac{M}{R}\left[u^2 - 2gR\right] - \frac{3Mg}{\sqrt{2}}\right\}$$

$$= 3\sqrt{2}Mg. \qquad \mathbf{2E1}$$

[END OF SECTION A SOLUTIONS]

Section B

B1. (a) $f(x) = e^{2x}\tan x \quad \Rightarrow \quad f'(x) = 2e^{2x}\tan x + e^{2x}\sec^2 x$ **1M,2E1**

(b) $g(x) = \dfrac{\cos 2x}{x^3} \quad \Rightarrow$

$$g'(x) = \frac{(-2\sin 2x)x^3 - (\cos 2x)(3x^2)}{x^6}$$ **1M,2E1**

$$= \frac{-2x\sin 2x - 3\cos 2x}{x^4}$$ **1**

The Product Rule correctly applied and executed would gain full marks.

B2. The term in a^6 must come from $\dfrac{1}{a^2} \times a^8$. Thus the term is

$$\binom{10}{2} \times \left(\frac{1}{a}\right)^2 \times (3a)^8$$ **1,1**

$$= 45 \times 6561 \times a^6$$ **1**

$$= 295245\,a^6$$ **1**

B3.

$$\frac{3x}{(x+1)^2} = \frac{A}{(x+1)^2} + \frac{B}{(x+1)}$$ **1M**

$$3x = A + B(x+1)$$

$$x = -1 \quad \Rightarrow \quad -3 = A$$ **1**

$$x = 0 \quad \Rightarrow \quad 0 = -3 + B \Rightarrow B = 3$$ **1**

$$\frac{3x}{(x+1)^2} = \frac{-3}{(x+1)^2} + \frac{3}{(x+1)}$$

$$\int \frac{3x}{(x+1)^2}\,dx = \int \frac{-3}{(x+1)^2} + \frac{3}{(x+1)}dx$$

$$= \frac{3}{x+1} + 3\ln(x+1) + c$$ **1,1**

B4.

$$\int y.dy = \int 9te^{3t}.dt$$ **1M,1**

$$\frac{y^2}{2} = 3t\int 3e^{3t}.dt - \int 3e^{3t}.dt$$ **1M,1**

$$= 3te^{3t} - e^{3t} + c$$ **1**

$$(0,2) \Rightarrow 2 = 0 - 1 + c \Rightarrow c = 3$$ **1**

$$y^2 = 2\left(3te^{3t} - e^{3t} + 3\right)$$

$$y = \sqrt{6te^{3t} - 2e^{3t} + 6}$$ **1**

B5. (a)

$$\det A = m(m - 0) - 1(0 + 2) + 1(0 - m) \qquad \mathbf{1}$$

$$= m^2 - m - 2 \qquad \mathbf{1}$$

$$= (m + 1)(m - 2) \Rightarrow m = -1,\ 2 \qquad \mathbf{1}$$

(b)

$$\left.\begin{matrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{matrix}\;\right|\;\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \Rightarrow \quad \left.\begin{matrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{matrix}\;\right|\;\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{matrix} \qquad \mathbf{1M}$$

$$\left.\begin{matrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{matrix}\;\right|\;\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{matrix} \ (\mathbf{1}) \quad \Rightarrow \quad \left.\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{matrix}\;\right|\;\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{matrix}$$

$$\left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{matrix}\;\right|\;\begin{matrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ -1 & 1 & 1 \end{matrix} \ (\mathbf{1}) \quad \Rightarrow \quad \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\;\right|\;\begin{matrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{matrix}$$

$$\text{i.e. } B^{-1} = \begin{pmatrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix} \qquad \mathbf{1}$$

(c)

$$\begin{pmatrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} \qquad \mathbf{1}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \mathbf{1}$$

[END OF SECTION B SOLUTIONS]