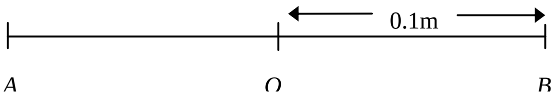
2009 Applied Mathematics

Advanced Higher – Mechanics

Finalised Marking Instructions

Advanced Higher Applied Mathematics 2009
Mechanics Solutions

A1.



(a) The maximum speed = $10 = \omega a$ **1M**

$$\Rightarrow \omega = \frac{10}{0{\cdot}1} = 100$$

$$\Rightarrow \text{Period of oscillation} = T = \frac{2\pi}{100} = \frac{\pi}{50}\ \text{s}$$ **1**

(b) Now $v^2 = \omega^2\left(a^2 - x^2\right)$ **1M**

$$= 100^2\left(0{\cdot}1^2 - 0{\cdot}05^2\right)$$

$$= 100^2 \times 0{\cdot}0075 = 75$$

When the particle is 5 cm from O, it has speed $8{\cdot}66\ \text{m s}^{-1}$. **1**

A2. Let the initial speed of the ball be $V\ \text{m s}^{-1}$.
The horizontal motion is given by

$$x = \left(V\cos\theta\right)t \qquad \Rightarrow \qquad 40 = \frac{4Vt}{5} \qquad \Rightarrow \qquad t = \frac{50}{V}.$$ **1**

The vertical motion is given by

$$y = \left(V\sin\theta\right)t - \frac{1}{2}gt^2$$ **1**

$$\Rightarrow \qquad 2 = \frac{3Vt}{5} - \frac{1}{2}gt^2.$$ **1**

Eliminating t gives

$$2 = 30 - \frac{2500g}{2V^2}$$ **1**

$$V^2 = \frac{2500g}{56}$$

$$V \approx 21\ \text{m s}^{-1}$$ **1**

A3. Firstly, $\mathbf{v}_p = 2t\mathbf{i} + 4\mathbf{j}$ **1**
and also $\mathbf{v}_Q = 2t\mathbf{i} - 2\cos 2\pi t\mathbf{j} + \mathbf{c}$ **1M, 1**
Given that when $t = 0$, $\mathbf{v}_Q = \mathbf{0}$ then $\mathbf{c} = 2\mathbf{j}$, so

$$\mathbf{v}_Q = 2t\mathbf{i} + 2\left(1 - \cos 2\pi t\right)\mathbf{j}$$ **1**

When the boats have the same velocity

$$2\left(1 - \cos 2\pi t\right) = 4 \Rightarrow \qquad \cos 2\pi t = -1$$

$$\Rightarrow \qquad t = 0{\cdot}5,\ 1{\cdot}5\ \text{s}$$ **1**

A4. (a) Starting with $\frac{dv}{dt} = a$ and integrating gives

$$v = at + c$$ **1M**

When $t = 0$, $v = u$ hence $c = u$ **1**

and so $v = u + at$ (*).

Thus we have $\frac{ds}{dt} = u + at$. Integrating gives

$$s = ut + \tfrac{1}{2}at^2 + k$$ **1**

Since $s = 0$ when $t = 0$ then $k = 0$ so $s = ut + \tfrac{1}{2}at^2$ (**) **1**

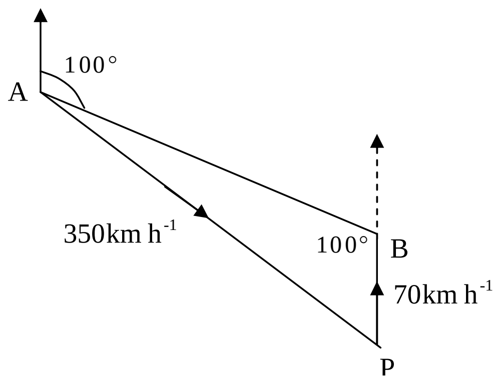
(b) From equation (*) $t = \frac{v - u}{a}$

and substitute into (**) to give $s = \frac{u(v - u)}{a} + \frac{(v - u)^2}{2a}$

$$\Rightarrow 2as = 2uv - 2u^2 + v^2 - 2uv + u^2 = v^2 - u^2$$ **2E1**

$$\Rightarrow v^2 = u^2 + 2as$$

A5. *Method 1 - velocities*



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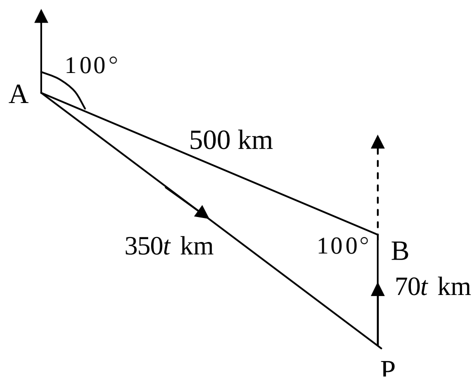
By the sine rule $\frac{\sin \angle PAB}{70} = \frac{\sin 100°}{350}$. **1**

Hence $\angle PAB = 11{\cdot}4°$.

So the required bearing is $111{\cdot}4°$ **1**

Method 2 - displacements



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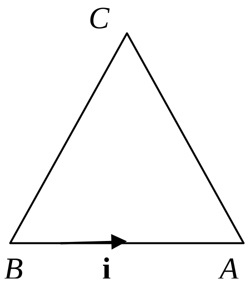
Let the time be t hours. Then

$$\frac{\sin \angle PAB}{70t} = \frac{\sin 100°}{350t} \Rightarrow \sin \angle PAB = \frac{\sin 100°}{5} \Rightarrow \angle PAB = 11{\cdot}4°$$ **1**

So the required bearing is $111{\cdot}4°$ **1**

A6.



Let the origin of a rectangular coordinate system be the point B and let $\mathbf{i}$ be the unit vector in the direction $\overrightarrow{BA}$. **1**

Then $\mathbf{v}_{A \to B} = -U\mathbf{i}$ and

$$\mathbf{v}_{B \to C} = U \cos 60°\mathbf{i} + U \sin 60°\mathbf{j} = \tfrac{1}{2}U\left(\mathbf{i} + \sqrt{3}\mathbf{j}\right)$$ **1**

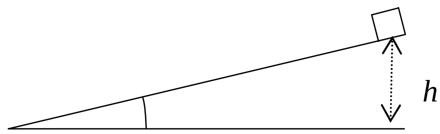
The impulse is given by

$$\mathbf{I} = m\left(\mathbf{v}_{B \to C} - \mathbf{v}_{A \to B}\right)$$ **1**

$$\Rightarrow \mathbf{I} = \tfrac{1}{2}mU\left(3\mathbf{i} + \sqrt{3}\mathbf{j}\right)$$ **1**

and the magnitude of the impulse is $|\mathbf{I}| = \sqrt{3}mU$ **1**

A7. Let m kg be the mass of the block, and a m s^{-1} its acceleration down the slope.



Then, $ma = mg \sin\theta \Rightarrow a = g \sin\theta$. **1M**

Let s be the distance travelled down the slope.

Now, $\sin\theta = \dfrac{h}{s} \Rightarrow s = \dfrac{h}{\sin\theta}$, **1**

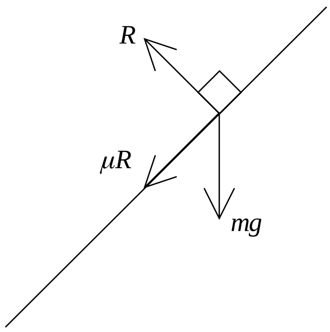
and also $s = ut + \tfrac{1}{2}at^2 = \tfrac{1}{2}(g \sin\theta)t^2$.

Hence $\dfrac{h}{\sin\theta} = \dfrac{g \sin\theta}{2}t^2$. **1**

$$\Rightarrow t^2 = \frac{2h}{g \sin^2\theta}.$$ **1**

$$\Rightarrow t = \sqrt{\frac{2h}{g \sin^2\theta}}.$$

A8.



Let R be the normal reaction force and μ the coefficient of friction between the cycle wheels and the track.

Resolving forces in the vertical direction

$$R\cos 45^\circ = \mu R\sin 45^\circ + mg$$ **1M**

$$\Rightarrow R(1-\mu) = \sqrt{2}mg \qquad (*)$$ **1**

Resolving in the horizontal direction

$$\frac{mv^2}{r} = R\sin 45^\circ + \mu R\cos 45^\circ$$ **1M**

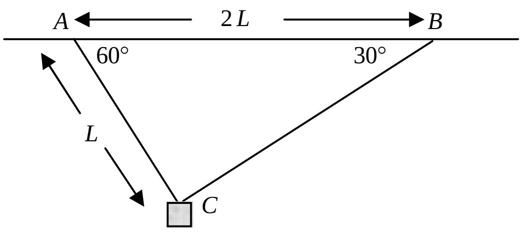
$$R(1+\mu) = 3\sqrt{2}mg \qquad (**)$$ **1**

Dividing equations (*) and (**)

$$\frac{1+\mu}{1-\mu} = 3$$ **1**

$$\Rightarrow \mu = \tfrac{1}{2}$$ **1**

A9. (a)



Let T_1 be the tension in AV and T_2 the tension in BC.
Resolving horizontally gives

$$T_1 \cos 60° = T_2 \cos 30°$$ **1M**

$$\Rightarrow T_1 = \sqrt{3}T_2 \qquad (*)$$ **1**

Resolving vertically gives

$$T_1 \sin 60° + T_2 \sin 30° = W$$

$$\Rightarrow \qquad \sqrt{3}T_1 + T_2 = 2W \qquad (*\ *)$$ **1**

Using (*) and (**) to eliminate T_1
we get $4T_2 = 2W \Rightarrow T_2 = 0{\cdot}5W$. **1**
So the tension in BC is $0{\cdot}5W$ newtons.

(b) Since $CB = \sqrt{3}L$, the extension of the string is $\left(\sqrt{3} - 1\right)L$. **1**

Using Hooke's law $T_2 = \dfrac{\lambda x}{L}$ **1M**

gives $0{\cdot}5W = \lambda\left(\sqrt{3} - 1\right)$

and the modulus of elasticity is $\dfrac{W}{2\left(\sqrt{3} - 1\right)}$. **1**

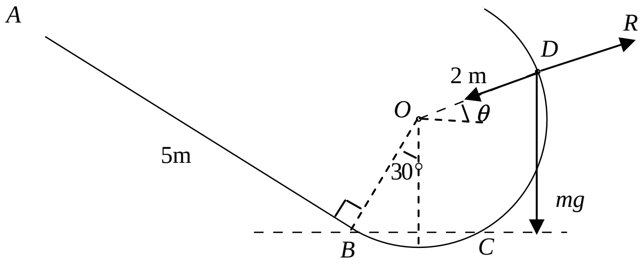
The elastic potential energy is $E = \dfrac{\lambda x^2}{2L}$, **1**

so

$$E = \frac{1}{2L}\frac{W}{2\left(\sqrt{3} - 1\right)}\left(\sqrt{3} - 1\right)^2 L^2$$ **1**

$$= \frac{1}{4}\left(\sqrt{3} - 1\right)LW \text{ joules}$$

A10.



(a) *Method 1 - work-energy*

Height of A above BC is $5 \sin 30° = 2{\cdot}5$ m.

Change in potential energy = $mg\Delta h = 0{\cdot}2 \times 9{\cdot}8 \times 2{\cdot}5 = 4{\cdot}9$ J **1M,1**

Work done overcoming friction = $Fs = 0{\cdot}08 \times 5 = 0{\cdot}4$ J **1**

At C, kinetic energy of sledge = $(4{\cdot}9 - 0{\cdot}4) = 4{\cdot}5$ J **1**

Method 2 - Newton's laws

Let a be the acceleration. Resolving parallel to AB

$$ma = mg \cos 60° - 0{\cdot}08$$ **1**

$$\Rightarrow a = 4{\cdot}9 - 0{\cdot}4 = 4{\cdot}5$$ **1**

$$v^2 = 0^2 + 2 \times 4{\cdot}5 \times 5 = 45$$ **1**

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2} \times 0{\cdot}2 \times 45 = 4{\cdot}5$$

At C, kinetic energy of sledge = $4{\cdot}5$ J **1**

(b) Let D be the point at which the sledge leaves the track so that, as the sledge approaches D, the normal force, R, tends to 0. **1M**

Let the speed of the sledge at D be v so that, at D,

$$mg \sin\theta = \frac{mv^2}{r} \quad \Rightarrow \quad v^2 = rg \sin\theta = 2g \sin\theta$$ **1M**

Hence at D, the kinetic energy of the sledge is $mg \sin\theta = 1{\cdot}96 \sin\theta$. **1**

Height of D above BC = $2 \sin\theta + 2 \cos 30° = 2 \sin\theta + 1{\cdot}732$

Potential energy at D is $mg\left(2 \sin\theta + \sqrt{3}\right) = 1{\cdot}96\left(2 \sin\theta + 1{\cdot}732\right)$ **1**

By the conservation of energy,

$$1{\cdot}96\left(2 \sin\theta + 1{\cdot}732\right) + 1{\cdot}96 \sin\theta = 4{\cdot}5$$ **1M**

$$3 \sin\theta + 1{\cdot}732 = 2{\cdot}296$$

$$\sin\theta = \frac{0{\cdot}564}{3} = 0{\cdot}188$$

Hence, the angle between OD and the horizontal is $10{\cdot}8°$. **1**

A11. (a) Whilst being towed, the equation of motion of the skier is

$$60\frac{dv}{dt} = 300 - 15v \qquad \textbf{1}$$

$$\Rightarrow \int dt = 4\int \frac{1}{20 - v}\, dv \qquad \textbf{1M}$$

$$\Rightarrow t = -4\ln|20 - v| + C \qquad \textbf{1}$$

When $t = 0, v = 0$, so $C = 4\ln 20$

and hence $t = 4\ln\left|\frac{20}{20 - v}\right|$ **1**

Rearranging gives $\frac{20}{20 - v} = e^{0{\cdot}25t}$ **1**

$$\Rightarrow 20 - v = 20e^{-0{\cdot}25t} \Rightarrow v = 20\left(1 - e^{-0{\cdot}25t}\right)$$

When $t = 6, v = 20\left(1 - e^{-1{\cdot}5}\right) = 15{\cdot}5 \text{ m s}^{-1}$.

The line BC passes through $(6, 15{\cdot}5)$ and $(10, 0)$. **1**

So an equation for BC is $v - 0 = -3{\cdot}9(t - 10)$

$$\Rightarrow v = 39 - 3.9t \qquad \textbf{1}$$

(b) Distance travelled between $t = 0$ and $t = 10$ is given by

$$s = \int_0^6 20\left(1 - e^{-0{\cdot}25t}\right)dt + \int_6^{10}(39 - 3.9t)\,dt \qquad \textbf{1M}$$

$$= 20\left[t + 4e^{-0{\cdot}25t}\right]_0^6 + \left[39t - 1.95t^2\right]_6^{10} \qquad \textbf{2E1}$$

$$= 20\left(6 + 4e^{-1{\cdot}5} - 4\right) + (390 - 195 - 234 + 70.2)$$

$$\approx 89.05\,\text{m} \qquad \textbf{1}$$

[END OF MECHANICS SOLUTIONS]

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Section B Solutions

B1.

$$\left(b - \frac{2}{b}\right)^5 = b^5 + 5b^4\left(-\frac{2}{b}\right) + 10b^3\frac{4}{b^2} + 10b^2\left(-\frac{8}{b^3}\right) + 5b\frac{16}{b^4} - \frac{32}{b^5}$$ **powers 1** **coeffs 1** **signs 1**

$$= b^5 - 10b^3 + 40b - \frac{80}{b} + \frac{80}{b^3} - \frac{32}{b^5}$$ **1**

B2.

$$u = \cos x \Rightarrow du = -\sin x\,dx,$$ **1**

$$x = 0 \Rightarrow u = 1; \qquad x = \tfrac{\pi}{3} \Rightarrow u = \tfrac{1}{2}$$ **1**

Hence

$$\int_0^{\pi/3} \cos^5 x \sin x\,dx = -\int_1^{\frac{1}{2}} u^5 du = \left[-\frac{1}{6}u^6\right]_1^{\frac{1}{2}}$$ **1**

$$= -\frac{1}{6}\frac{1}{64} + \frac{1}{6} = \frac{21}{128}\,(\approx 0.164)$$ **1**

OR

$$\int_0^{\pi/3} \cos^5 x \sin x\,dx = \left[-\frac{1}{6}\cos^6 x\right]_0^{\pi/3}$$ **3E1**

$$= -\frac{1}{6}\frac{1}{64} + \frac{1}{6} = \frac{21}{128}\,(\approx 0.164)$$ **1**

B3.

$$x = t^2 + 1 \Rightarrow \frac{dx}{dt} = 2t$$

$$y = 1 - 3t^3 \Rightarrow \frac{dy}{dt} = -9t^2$$ **1**

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$ **M1**

$$= \frac{-9t^2}{2t} = \frac{-9t}{2}$$

$$= -9 \text{ when } t = 2.$$ **1**

Point of contact is $x = 5, y = -23$. **1**

Equation of tangent is

$$(y + 23) = -9(x - 5)$$ **1**

$$y + 23 = -9x + 45$$

$$y + 9x = 22$$

B4.

$$\det\begin{pmatrix}1 & 1 & 0\\ 0 & k-2 & -1\\ 1 & 2 & k\end{pmatrix} = 1\det\begin{pmatrix}k-2 & -1\\ 2 & k\end{pmatrix} - 1\det\begin{pmatrix}0 & -1\\ 1 & k\end{pmatrix} + 0 \quad \textbf{M1,1}$$

$$= (k-2)k + 2 - (0+1) \quad \textbf{1}$$

$$= k^2 - 2k + 1 = (k-1)^2 = 0.$$

Hence the matrix does not have an inverse when $k = 1$. **1**

B5.

$$t\frac{dx}{dt} - 2x = 3t^2$$

$$\frac{dx}{dt} - \frac{2}{t}x = 3t \quad \textbf{1}$$

Integrating factor: $\int -\frac{2}{t}dt = -2\ln t = \ln t^{-2}$ so IF = t^{-2}. **M1,1**

$$\frac{1}{t^2}\frac{dx}{dt} - \frac{2}{t^3}x = \frac{3}{t}$$

$$\frac{x}{t^2} = \int\frac{3}{t}dt \quad \textbf{1}$$

$$= 3\ln t + c$$

$$x = t^2(3\ln t + c) \quad \textbf{1}$$

$$(1,1) \Rightarrow c = 1 + 0$$

$$x = t^2(1 + 3\ln t) \quad \textbf{1}$$

B6.

$$f(x) = x\tan 2x$$

$$f'(x) = \tan 2x + 2x\sec^2 2x \quad \textbf{M1,1}$$

$$f''(x) = 2\sec^2 2x + 2\sec^2 2x + 2x(4\sec 2x(\sec 2x\tan 2x)) \quad \textbf{2E1}$$

$$= 4\sec^2 2x + 8x\sec^2 2x\tan 2x \quad \textbf{1}$$

$$= 4\sec^2 2x(1 + 2x\tan 2x).$$

$$\int_0^{\pi/6}\frac{1 + 2x\tan 2x}{\cos^2 2x}dx = \frac{1}{4}\int_0^{\pi/6}4\sec^2 2x(1 + 2x\tan 2x)dx \quad \textbf{1,1}$$

$$= \frac{1}{4}\left[\tan 2x + 2x\sec^2 2x\right]_0^{\pi/6} \quad \textbf{1}$$

$$= \frac{1}{4}\left[\sqrt{3} + \frac{\pi}{3}2^2\right] \quad \textbf{1}$$

$$= \frac{\sqrt{3}}{4} + \frac{\pi}{3}.$$

[END OF SECTION B SOLUTIONS]