2008 Applied Mathematics

Advanced Higher – Mechanics

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values/algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question B6, M1 means a method mark for separating the variables. E is shorthand for error. For example, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Applied Mathematics 2008

Section A − Mechanics

A1.

$$v = 3t(2 - t) = 6t - 3t^2 \Rightarrow \frac{dv}{dt} = 6 - 6t$$

$v_{\max}$ occurs when $\frac{dv}{dt} = 0 \Rightarrow 6 - 6t = 0 \Rightarrow t = 1.$ **1**

$$s = \int (6t - 3t^2)\,dt = 3t^2 - t^3 + c$$ **1**

At $t = 0, s = 3 \Rightarrow c = 3.$ Hence,

$$s = 3 + 3t^2 - t^3$$ **1**

At $t = 1$, $s = 5$ so the distance from O is 5 m. **1**

A2. Using $s = ut + \frac{1}{2}at^2$, with $s = 120$, $u = 2$, $a = \frac{1}{4}$, $t = T$ **1**

$$120 = 2T + \tfrac{1}{8}T^2$$ **1**

$$T^2 + 16T - 960 = 0$$

$$(T - 24)(T + 40) = 0$$ **1**

Hence $T = 24$ since $T > 0$. **1**

Using $v = u + at$ **1**

$$v = 2 + 24 \times \tfrac{1}{4} = 8 \text{ m s}^{-1}$$ **1**

Alternative ending

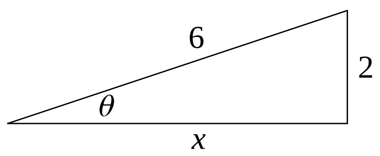
$$v^2 = u^2 + 2as$$ **1**

$$v^2 = 2^2 + 2 \times \tfrac{1}{4} \times 120$$ **1**

$$= 64$$ **1**

$$v = 8 \text{ m s}^{-1}$$ **1**

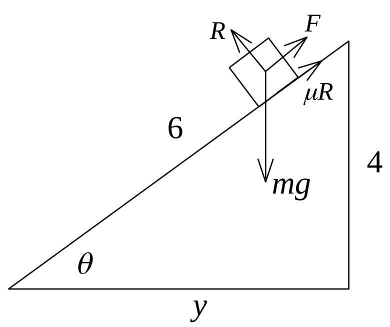
A3.



(a) By Pythagoras,

$x = \sqrt{32} = 4\sqrt{2}$ **1**

$\mu = \tan\theta = \dfrac{2}{4\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$ **1**



(b) By Pythagoras, $y = \sqrt{20} = 2\sqrt{5}$.

Resolving at right angles to the plane,

$R = mg\cos\theta = 2g \times \dfrac{2\sqrt{5}}{6}$ **1**

$= \dfrac{2\sqrt{5}g}{3} = 14{\cdot}61 \text{ N}$

Friction $= \mu R = \dfrac{1}{2\sqrt{2}} \times 14{\cdot}61$ **1**

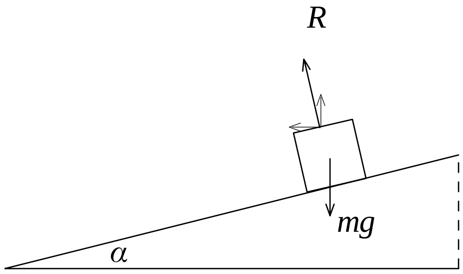
$= 5{\cdot}17 \text{ N}$

The component of the weight acting down the plane has magnitude

$$mg\sin\theta = 2g \times \frac{2}{3} = 13{\cdot}07 \text{ N}$$

Hence $F \approx 13{\cdot}07 - 5{\cdot}17 = 7{\cdot}90 \text{ N}$ **1**

A4.



$$R \sin\alpha = \frac{mv^2}{r} \qquad \mathbf{1}$$

$$R \cos\alpha = mg \qquad \mathbf{1}$$

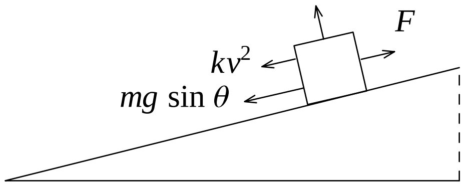
Hence

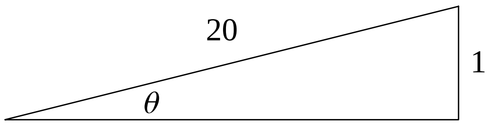
$$\tan\alpha = \frac{v^2}{gr}$$

$$v^2 = gr\tan\alpha \qquad \mathbf{1}$$

so that $v = \sqrt{gr\tan\alpha}$

A5.





Resolving parallel to the plane, by Newton II,

$$ma = F - mg\sin\theta - kv^2 \qquad \mathbf{1}$$

$$100 \times 0{\cdot}05 = F - 100 \times 9{\cdot}8 \times \frac{1}{20} - 4k \qquad \mathbf{1}$$

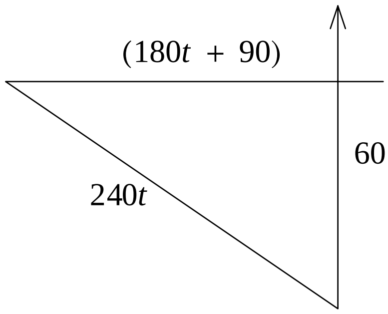
Also, $P = Fv$ **1**

so that $F = \frac{120}{2} = 60$ N **1**

Hence, $4k = 60 - 5 - 5 \times 9{\cdot}8 = 6$ **1**

so that $k = \frac{3}{2}$. **1**

A6. (a) Let the time to the expected collision be t hours.



By Pythagoras,

$$(240t)^2 = (180t + 90)^2 + 60^2 \qquad \mathbf{1}$$

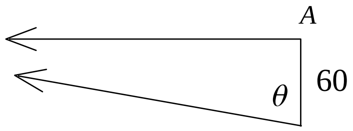
$$\Rightarrow 57600t^2 = 32400t^2 + 32400t + 8100 + 3600$$

$$252t^2 - 324t - 117 = 0 \qquad \mathbf{1}$$

$$t = \frac{324 \pm \sqrt{324^2 + 117936}}{504} \qquad \mathbf{1}$$

$$= 1{\cdot}58$$

Time to collision = 1 hour 35min **1**



Alternative for first two marks:

Let $\mathbf{r}_1 = (90 + 180t)\,\mathbf{i}$ and

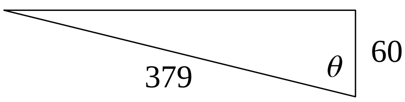
$\mathbf{r}_2 = (240\sin\theta)t\mathbf{i} + \big((240\cos\theta)t - 60\big)\mathbf{j}$ **1**

Collision if: $(240\cos\theta)t - 60 = 0 \Rightarrow \cos\theta = \frac{1}{4t} \Rightarrow \sin^2\theta = 1 - \frac{1}{16t^2}$; and $(240\sin\theta)t = (90 + 180t)$ **1**

$\Rightarrow 8\sin\theta\, t = 3 + 6t \Rightarrow \sin^2\theta = \left(\frac{3+6t}{8t}\right)^2 \Rightarrow \left(\frac{3+6t}{8t}\right)^2 = \left(1 - \frac{1}{16t^2}\right) \Rightarrow$

$64t^2 - 4 = 9 + 36t + 36t^2$, etc

(b)



$$\cos\theta = \frac{60}{379} \Rightarrow \theta = 80{\cdot}9° \qquad \mathbf{1}$$

The bearing is 279·1°. **1**

A7. (a) At terminal velocity, $80 + \frac{1}{4}v^2 = 100 \times 9{\cdot}8$, **1**

$$\Rightarrow \frac{1}{4}v^2 = 900 \Rightarrow v^2 = 3600 \Rightarrow v = 60 \text{ m s}^{-1}$$ **1**

(b) 75% of terminal velocity = 45 m s^{-1}

$$\Rightarrow ma = mg - \left(80 + \tfrac{1}{4}v^2\right)$$ **1**

$$\Rightarrow a = 9{\cdot}8 - \left(0{\cdot}8 + 0{\cdot}0025v^2\right)$$

$$\Rightarrow \frac{dv}{dt} = \left(9 - 0{\cdot}0025v^2\right)$$ **1**

$$\Rightarrow t = \int_0^{45} \frac{dv}{\left(9 - 0{\cdot}0025v^2\right)}$$ **1 structure**
1 for limits

$$= \left[\frac{1}{2 \times 3 \times 0{\cdot}05} \ln\left(\frac{3 + 2{\cdot}25}{3 - 2{\cdot}25}\right)\right] - [0] = \left[\frac{10}{3}\ln 7\right] (\approx 6{\cdot}49 \text{ s})$$ **1**

A8. (a) Using $v^2 = \omega^2\left(a^2 - x^2\right)$ we have **1**

$$\frac{\pi^2}{3} = \omega^2\left(a^2 - 1\right) \ldots (1) \text{ and } \frac{\pi^2}{9} = \omega^2\left(a^2 - 3\right) \ldots (2)$$ **1**

Dividing (1) by (2), $\dfrac{a^2 - 1}{a^2 - 3} = 3.$ **1**

$$\Rightarrow a^2 - 1 = 3a^2 - 9 \Rightarrow 2a^2 = 8 \Rightarrow a^2 = 4$$ **1**

$$\Rightarrow a = 2 \text{ since } a > 0$$

With $a = 2$, from (1), $\omega^2 = \dfrac{\pi^2}{9} \Rightarrow \omega = \dfrac{\pi}{3}$ and period $= \dfrac{2\pi}{\omega} = 6\text{ s}$ **1**

(b) The distance of the target from O is $x = 2\sin\frac{\pi}{3}t$. **1**

When T reaches B, $2\sin\dfrac{\pi}{3}t = 2 \Rightarrow \dfrac{\pi}{3}t = \dfrac{\pi}{2} \Rightarrow t = \dfrac{3}{2}$. **1**

When T was at P, $t = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$. **1**

So

$$x = 2\sin\left(\frac{\pi}{3} \times \frac{3}{4}\right) = 2\sin\frac{\pi}{4} = 2 \times \frac{1}{\sqrt{2}}$$ **1**

Hence $PB = 2 - \dfrac{2}{\sqrt{2}} = 2 - \sqrt{2}$ m. **1**

9. (a) At A, $E_p = mgh$, hence

$$E_p = 0{\cdot}1g \times 2 = 0{\cdot}2g \quad \mathbf{1}$$

At O,

$$E_k = \tfrac{1}{2}mv^2 = \tfrac{1}{2} \times 0{\cdot}1 \times v^2 = 0{\cdot}2g \quad \mathbf{1}$$

$$\Rightarrow v^2 = 4g \Rightarrow v = 2\sqrt{g} \quad \mathbf{1}$$

The time taken to fall a vertical distance of 1 m is given by

$$s = \frac{1}{2}gt^2 \Rightarrow gt^2 = 2 \Rightarrow t^2 = \frac{2}{g} \Rightarrow t = \sqrt{\frac{2}{g}} \quad \mathbf{1}$$

Horizontal distance travelled: $a = 2\sqrt{g} \times \sqrt{\tfrac{2}{g}} = 2\sqrt{2}$ m **1**

(b) On the track, $y = x(x+1) \Rightarrow \dfrac{dy}{dx} = 2x + 1$ **1**

When $x = 0$, slope $= 1 \Rightarrow \tan\theta = 1$ **1**

As in (a), $E_p = mgh = 0{\cdot}2g$ and $E_k = \tfrac{1}{2}mv^2 = 0{\cdot}2g \Rightarrow v^2 = 4g$ **1**

To determine maximum height:

At O, the vertical component of the velocity is $\sqrt{4g}\cos 45° = \sqrt{2g}$. **1**

Hence, $v^2 = u^2 + 2as \Rightarrow 0 = 2g - 2gs \Rightarrow s = 1$ m **1**

Thus the height reached above ground level = $1 \text{ m} + 1 \text{ m} = 2 \text{ m}$. **1**

A10. (a) By the Conservation of Momentum,

$$mU\mathbf{i} + MU\mathbf{j} = (m+M)(V\cos\theta\mathbf{i} + V\sin\theta\mathbf{j}) \quad \mathbf{2E1}$$

In component form,

$$(m+M)V\cos\theta = mU \quad \ldots (1)$$
$$(m+M)V\sin\theta = MU \quad \ldots (2) \quad \mathbf{1}$$

Dividing (2) by (1) $\dfrac{\sin\theta}{\cos\theta} = \dfrac{M}{m}$ **1**

$$\Rightarrow \tan\theta = \frac{M}{m}.$$

Squaring (1) and (2) and adding,

$$(m+M)^2V^2(\cos^2\theta + \sin^2\theta) = (m^2 + M^2)U^2 \quad \mathbf{1,1}$$

$$\Rightarrow V^2 = \frac{(m^2 + M^2)U^2}{(m+M)^2}$$

(b) Since $\tan\theta = 2$, $M = 2m$ **1**

$$\Rightarrow V^2 = \frac{(m^2 + 4m^2)U^2}{(3m)^2} = \frac{5}{9}U^2 \quad \mathbf{1}$$

Loss in K.E.

$$= \tfrac{1}{2}MU^2 + \tfrac{1}{2}mU^2 - \tfrac{1}{2}(M+m)V^2 \quad \mathbf{1}$$

$$= \tfrac{3}{2}mU^2 - \tfrac{1}{2} \times 3m \times \tfrac{5}{9}U^2 = \tfrac{3}{2}mU^2\left(1 - \tfrac{5}{9}\right)$$

$$= \tfrac{2}{3}mU^2 \quad \mathbf{1}$$

Section B − Mathematics for Applied Mathematics

B1. (a) $AB = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 15 \\ 12 & 3 \end{pmatrix}$ **1**

(b)

$$4C + D = 4\begin{pmatrix} x & 2 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4x & 8 \\ 0 & 4y \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4x + 2 & 15 \\ 12 & 4y - 1 \end{pmatrix}$$ **2E1**

(c) $\begin{pmatrix} 8 & 15 \\ 12 & 3 \end{pmatrix} = \begin{pmatrix} 4x + 2 & 15 \\ 12 & 4y - 1 \end{pmatrix} \Rightarrow x = 1{\cdot}5;\ y = 1$ **1,1**

B2.

$$y = e^{2x}\cos x$$

$$\frac{dy}{dx} = 2e^{2x}\cos x + e^{2x}(-\sin x)$$ **1M, 2E1**

$$= 2e^{2x}\cos x - e^{2x}\sin x$$

B3.

$$\frac{4x - 3}{x(x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}$$ **1M**

$$4x - 3 = A(x^2 + 3) + x(Bx + C)$$

$$= (A + B)x^2 + Cx + 3A$$

$$A = -1,\ B = 1,\ C = 4$$ **2E1**

$$\frac{4x - 3}{x(x^2 + 3)} = \frac{-1}{x} + \frac{x + 4}{x^2 + 3}$$ **1**

B4. (a)

$$\int \ln x\, dx = \int \ln x.1\, dx$$ **1**

$$= x\ln x - \int x.\frac{1}{x}dx$$

$$= x\ln x - x + c$$ **1**

(b) Volume of a solid of revolution $= \int \pi y^2 dx$ **1**

$$\text{Volume of goblet} = \pi\int_1^{10}\left(2\sqrt{\ln x}\right)^2 dx$$

$$= 4\pi\int_1^{10}\ln x\, dx$$ **1**

$$= 4\pi\left[x\ln x - x\right]_1^{10}$$ **1**

$$= 4\pi\left[(10\ln 10 - 10) - (0 - 1)\right]$$

$$= 4\pi\left[10\ln 10 - 9\right]\ (\approx 176{\cdot}25 \text{ is acceptable})$$ **1**

B5. (a)

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \mathbf{1}$$

$$\sum_{r=1}^{n}\left(6r^2 - r\right) = 6\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r \qquad \mathbf{1}$$

$$= n(n+1)(2n+1) - \frac{1}{2}n(n+1) \qquad \mathbf{1}$$

$$= \tfrac{1}{2}n(n+1)(4n+1)$$

(b)

$$\sum_{r=5}^{10}\left(6r^2 - r\right) = \sum_{r=1}^{10}\left(6r^2 - r\right) - \sum_{r=1}^{4}\left(6r^2 - r\right) \qquad \mathbf{1}$$

$$= \tfrac{1}{2} \times 10 \times 11 \times 41 - \tfrac{1}{2} \times 4 \times 5 \times 17$$

$$= 2085 \qquad \mathbf{1}$$

B6. (a)

$$\int \frac{1}{T - 22}\,dT = \int k\,dt \qquad \mathbf{M1,1}$$

$$\ln(T - 22) = kt + c \qquad \mathbf{1}$$

$$T - 22 = e^{kt + c} \qquad \mathbf{1}$$

$$T = Ae^{kt} + 22$$

(b)

$$82 = Ae^{k \times 0} + 22 \Rightarrow A = 60 \qquad \mathbf{1}$$

$$62 = 60e^{k \times 5} + 22 \Rightarrow 40 = 60e^{5k} \Rightarrow e^{5k} = \frac{2}{3} \qquad \mathbf{1}$$

$$\Rightarrow \ln\frac{2}{3} = 5k \Rightarrow k = \frac{1}{5}\ln\frac{2}{3} \qquad \mathbf{1}$$

$$T = 60e^{\frac{1}{5}\left(\ln\frac{2}{3}\right)t} + 22 \qquad \left(T = 60e^{-0\cdot08t} + 22 \text{ is acceptable}\right) \qquad \mathbf{1}$$

$$T = 60e^{2\ln\frac{2}{3}} + 22$$

$$= 60e^{\ln\frac{4}{9}} + 22$$

$$= 60 \times \tfrac{4}{9} + 22 = 48\tfrac{2}{3} \qquad \left(\approx 48\cdot7 \text{ is acceptable}\right) \qquad \mathbf{1}$$

[END OF MARKING INSTRUCTIONS]