X204/701

NATIONAL QUALIFICATIONS 2009

MONDAY, 25 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Mechanics

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics
3. **Full credit will be given only where the solution contains appropriate working.**

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Section A (Mechanics 1 and 2)

Marks

Answer all the questions.

Candidates should observe that $g\,\text{m s}^{-2}$ denotes the magnitude of the acceleration due to gravity.
Where appropriate, take its magnitude to be $9{\cdot}8\,\text{m s}^{-2}$.

A1. A particle is projected from O at time $t = 0$ and performs simple harmonic motion with O as the centre of the oscillation. The amplitude is 10 cm and the speed of projection is $10\ \text{m s}^{-1}$. Calculate:

(*a*) the period of the oscillation; **2**

(*b*) the speed of the particle when it is 5 cm from O. **2**

A2. On a horizontal cricket field, a batsman strikes a cricket ball towards a fielder standing 40 metres away. The ball is projected from ground level at an angle $\theta°$ to the horizontal, where $\tan\theta° = \frac{3}{4}$, and is caught by the fielder when it is 2 metres above the ground, without having hit the ground first.

Calculate the speed with which the ball leaves the bat. **5**

A3. The position of a model boat P, relative to a rectangular coordinate system with origin O, is given by

$$\mathbf{r}_P = t^2\mathbf{i} + 4t\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the Ox and Oy directions respectively, t is the time measured in seconds and distances are measured in metres.

The acceleration of a second boat Q is given by

$$\mathbf{a}_Q = 2\mathbf{i} + (4\pi\ \sin 2\pi t)\,\mathbf{j}.$$

Given that boat Q is initially at rest, find the first two times when the boats have the same velocity. **5**

A4. A particle moves in a straight line from the origin with initial velocity $u\mathbf{i}$ and uniform acceleration $a\mathbf{i}$, where $\mathbf{i}$ is a unit vector in a fixed direction. After time t, the velocity of the particle is $v\mathbf{i}$ and the displacement from the origin is $s\mathbf{i}$.

(*a*) Using calculus, show that $v = u + at$ and that $s = ut + \frac{1}{2}at^2$. **4**

(*b*) Hence show that $v^2 = u^2 + 2as$. **2**

A5. An aircraft is flying from airport A to airport B, which is 500 kilometres from A on a bearing of 100°. A wind, with speed 70 kilometres per hour, is blowing from the south throughout the flight. The speed of the aircraft in still air is 350 kilometres per hour.

Calculate the bearing on which the aircraft should fly to reach B. **3**

Marks

A6. Football players Ali, Billy and Carrie each stand on a different vertex of an equilateral triangle. Ali passes a football of mass m kg which reaches Billy with speed U m s^{-1}. Billy diverts the ball to Carrie with speed U m s^{-1}. Assume that the ball travels between players along the ground in a straight line.

Find an expression, in terms of m and U, for the magnitude of the impulse given to the ball by Billy. **5**

A7. A block is released from rest at the top of a smooth plane which is inclined at angle θ to the horizontal. Show that the time, in seconds, taken for the block to reach the bottom of the plane is given by

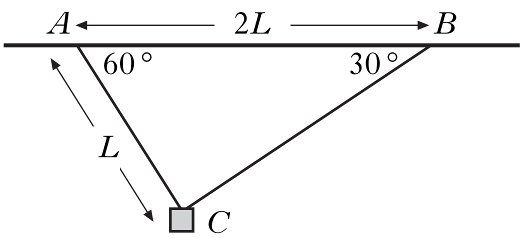
$$\sqrt{\frac{2h}{g\sin^2\theta}}$$

where h metres is the vertical distance between the top and the bottom of the plane. **4**

A8. The bend on a racetrack is semi-circular with radius r metres and is banked at 45° to the horizontal. A cyclist takes the bend with speed $\sqrt{3gr}$ m s^{-1} and is on the point of sliding **up** the banked track.

Calculate the coefficient of friction between the cycle wheels and the surface of the racetrack. **6**

A9. A small block of weight W newtons is suspended in equilibrium from a horizontal ceiling by two strings AC and BC as shown in the diagram where $\angle BAC = 60°$ and $\angle ABC = 30°$.



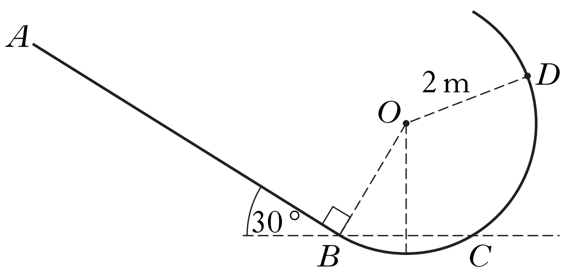
The distance between A and B is $2L$ metres and the string AC has length L metres.

(a) Show that the magnitude of the tension in string BC is $0{\cdot}5\,W$ newtons. **4**

(b) The string BC is elastic with natural length L metres. Calculate the modulus of elasticity of BC and show that the elastic potential energy stored in the string is $\frac{1}{4}(\sqrt{3}-1)LW$ joules. **5**

[Turn over

A10. A track consists of a rough, straight section AB which is inclined at an angle of $30°$ to the horizontal. The section AB is tangential to a semi-circular section of smooth track lying in the same vertical plane as AB. The centre of the semi-circle is at O and the radius is 2 m. The point C is at the same horizontal level as B.



A miniature sledge of mass 0·2 kg is released from rest at A. The section AB is 5 m long and provides a constant resistive force of 0·08 N to the motion of the sledge. The sledge continues on the smooth curved section, losing contact with the track at a point D. Calculate:

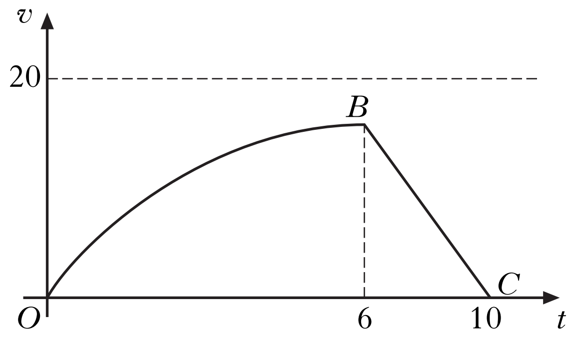
(*a*) the kinetic energy of the sledge at C; **4**

(*b*) the angle between OD and the horizontal. **6**

A11. A water skier and all his equipment have a total mass of 60 kg. He is towed by a speedboat which generates a tension of 300 newtons in the inelastic towing rope. The surface of the water, which should be assumed to be flat, creates a resistive force of $15\,v$ newtons, when the speed of the skier is v m s^{-1}.

The skier starts from rest and after 6 seconds reaches the bottom of a ramp and immediately releases the towing rope. Under uniform deceleration, he comes to rest on the ramp in a further 4 seconds. The speed/time graph of the skier is shown below, where t is the time measured in seconds from the start of the motion.

(It should be assumed that the ramp is designed so that there is no change in speed when the skier moves from the water to the ramp.)



(*a*) Show that the equation of the curve OB is $v = 20(1 - e^{-0{\cdot}25t})$ and that the equation of BC can be approximated by $v = 39 - 3{\cdot}9\,t$. **5,2**

(*b*) Calculate the total distance travelled by the water skier. **4**

[*END OF SECTION A*]

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

B1. Obtain the binomial expansion of $\left(b - \frac{2}{b}\right)^5$ and simplify the expression. **4**

B2. Obtain $\int_0^{\pi/3} \cos^5 x \sin x \, dx$ by using the substitution $u = \cos x$ or otherwise. **4**

B3. A particle moves along a curve in the x-y plane. The curve is defined by the parametric equations

$$x = t^2 + 1, \qquad y = 1 - 3t^3,$$

where t is the time elapsed since the start.

Find $\frac{dy}{dx}$ in terms of t and hence obtain an equation of the tangent to the curve when $t = 2$. **5**

B4. Determine k such that the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & k-2 & -1 \\ 1 & 2 & k \end{pmatrix}$ does not have an inverse. **4**

B5. An industrial scientist finds that the differential equation

$$t\frac{dx}{dt} - 2x = 3t^2$$

models a production process.

Find the general solution of the differential equation. **5**

Hence find the particular solution given $x = 1$ when $t = 1$. **1**

[Turn over for Question B6 on *Page six*

Marks

B6. Given $f(x) = x \tan 2x$ for $-\frac{\pi}{4} < x < \frac{\pi}{4}$, obtain an expression for $f'(x)$ and show that $f''(x) = 4\sec^2 2x\,(1 + 2x \tan 2x)$. **2,3**

Hence find the exact value of $\int_0^{\pi/6} \frac{1+2x \tan 2x}{\cos^2 2x}\,dx$. **4**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]