X204/701

NATIONAL
QUALIFICATIONS
2008

THURSDAY, 29 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Mechanics

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

    Section A assesses the Units Mechanics 1 and 2
    Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**

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Section A (Mechanics 1 and 2)

Marks

Answer all the questions.

Candidates should observe that $g\,\mathrm{m\,s^{-2}}$ denotes the magnitude of the acceleration due to gravity.

Where appropriate, take its magnitude to be $9{\cdot}8\,\mathrm{m\,s^{-2}}$.

A1. A particle has velocity $3t(2-t)\mathbf{j}$ where $\mathbf{j}$ is the unit vector in the direction of motion. The time t is measured in seconds from the start of the motion and the displacement is measured in metres. Initially the particle is at the point with position vector $3\mathbf{j}$ relative to the origin O. Calculate the distance of the particle from O when the velocity is a maximum. **4**

A2. Towards the end of a long-distance race, Tessa is running at a uniform speed of $2\,\mathrm{m\,s^{-1}}$. When she is 120 m from the finishing line she starts to increase her speed. In doing so, she accelerates uniformly at $0{\cdot}25\,\mathrm{m\,s^{-2}}$ for T seconds until she crosses the finishing line.

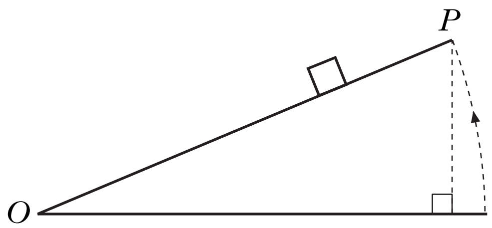
Show that T satisfies the equation

$$T^2 + 16T - 960 = 0$$

and hence find her speed as she crosses the finishing line. **6**

A3. A rough ramp OP of length 6 m is hinged at O. A point P at the other end is able to move about O in a vertical plane as illustrated in the diagram.

A small box of mass 2 kg is in equilibrium on the ramp.



(*a*) When P is 2 m above the horizontal plane through O, the box is on the point of sliding down the ramp. Calculate the coefficient of friction between the box and the ramp. **2**

(*b*) P is now raised to a height of 4 m above the horizontal plane. A force of F newtons, applied to the box and acting parallel to the ramp, is just sufficient to prevent the box from sliding down the ramp. Calculate the magnitude of F. **3**

A4. A bend on a smooth racing track forms an arc of a circle of radius r metres. The track is banked at an angle α to the horizontal. A car takes the bend at speed $v\,\mathrm{m\,s^{-1}}$ with no tendency to move either up or down the track.

Express v in terms of α, r and g. **3**

A5. Ben is cycling up a straight road which is inclined at an angle θ to the horizontal where $\sin\theta = \frac{1}{20}$. The combined mass of Ben and the cycle is 100 kg. The resistance to the motion from non-gravitational forces is a force of magnitude kv^2 newtons, where $v\,\text{m s}^{-1}$ is the speed of the cycle and k is a constant.

When Ben is cycling up the road at $2\,\text{m s}^{-1}$, his acceleration is $0{\cdot}05\,\text{m s}^{-2}$ and the rate at which he is working is 120 W.

Calculate the value of the constant k. **6**

A6. At 12 noon, an aircraft is above a point A and is flying due West at a uniform speed of $180\,\text{km h}^{-1}$. Thirty minutes later, a second aircraft, which is flying at exactly the same height as the first with a uniform speed of $240\,\text{km h}^{-1}$, is 60 km due south of A. The aircraft are on a collision course.

(*a*) Calculate the time when the collision would take place if no evasive action were taken. **4**

(*b*) Calculate the bearing on which the second aircraft is travelling. **2**

A7. A parachutist with all her equipment has a total mass of 100 kg. She jumps from a helicopter which is hovering at a constant height and falls vertically under gravity. She experiences a resistive force of magnitude $(80 + 0{\cdot}25\,v^2)$ newtons, where $v\,\text{m s}^{-1}$ is her speed.

(*a*) Calculate the terminal velocity of the parachutist. **2**

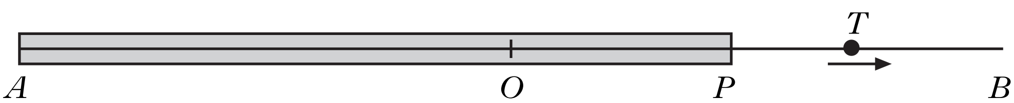
(*b*) Calculate the time for the parachutist to achieve 75% of her terminal velocity. **5**

[You may use the result that $\displaystyle\int \frac{1}{a^2 - b^2x^2}\,dx = \frac{1}{2ab}\ln\left(\frac{a+bx}{a-bx}\right)$.]

A8. In a fairground game, a small target T executes simple harmonic motion about a point O with extreme points A and B. When the target is 1 metre from O, its speed is $\frac{\pi}{\sqrt{3}}\,\text{m s}^{-1}$ and when it is $\sqrt{3}$ metres from O its speed is $\frac{\pi}{3}\,\text{m s}^{-1}$.

(*a*) Show that the amplitude of the motion is 2 metres and calculate the period of the oscillation. **5**

(*b*) A player has to shoot at the target, but it is only visible to the player when it is to the right of the point P as shown in the diagram.



Given that the target takes 0·75 seconds to move from P to B, calculate the distance PB. **5**

Marks

A9. A ball of mass 0·1 kg is released from a point A at the top of a smooth runway AO. The point O is 1 metre above ground level and, when the ball reaches O, it falls to the ground under the action of gravity.

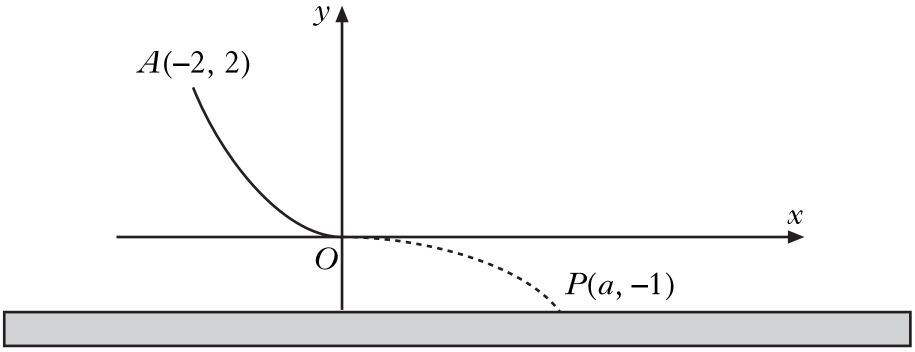


Figure 1

(a) Relative to the axes shown in Figure 1, the runway is modelled by the curve $y = \frac{1}{2}x^2$. The point A is (–2, 2) and O is the origin. The ball reaches the ground at $P(a, -1)$. Calculate the value of a. **5**

(b) The track is modified to run between the same two points A and O with the shape modelled by the equation $y = x(x + 1)$ as shown in Figure 2.

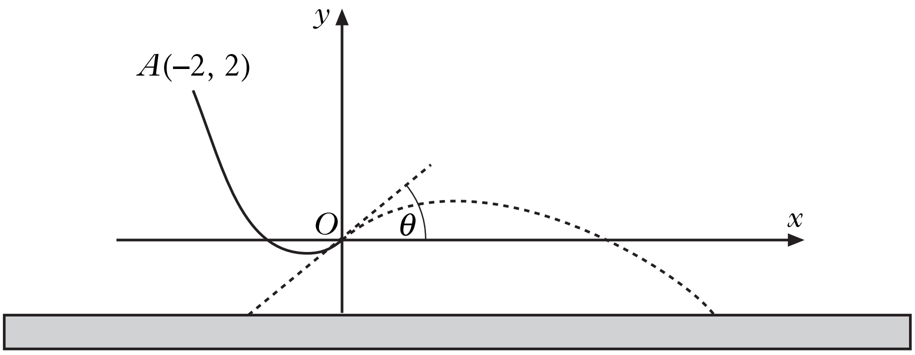


Figure 2

Calculate the maximum height above ground level attained by the ball after it has passed through O. **6**

Marks

A10. Ice-dancers, Alice and Bob, are skating on a smooth ice rink.

Alice has mass m kg and is moving with a constant velocity $U\mathbf{i}$, where $\mathbf{i}$ is the unit vector in the direction of the x-axis and U is measured in m s^{-1}.

Bob has mass M kg and is moving with a constant velocity $U\mathbf{j}$, where $\mathbf{j}$ is the unit vector in the direction of the y-axis.

The dancers collide and subsequently move off together with speed V m s^{-1} in a direction which makes an angle θ with the x-axis.

(*a*) Use conservation of momentum to show that

$$\tan\theta = \frac{M}{m} \qquad \text{and} \qquad V^2 = \frac{(m^2 + M^2)U^2}{(m+M)^2}.$$ **6**

(*b*) Given that $\tan\theta = 2$, find an expression for the kinetic energy lost in the collision in terms of m and U. **4**

[*END OF SECTION A*]

[Turn over for Section B on *Pages six* and *seven*

Section B (Mathematics for Applied Mathematics)

Answer all the questions.

Marks

B1. Given that A, B, C and D are square matrices where:

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix} \quad C = \begin{pmatrix} x & 2 \\ 0 & y \end{pmatrix} \quad D = \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix}$$

(*a*) Find AB. **1**

(*b*) Express $4C + D$ as a single matrix. **2**

(*c*) Given that $AB = 4C + D$, find the values of x and y. **2**

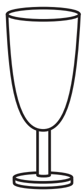
B2. Given that $y = e^{2x}\cos x$, find $\dfrac{dy}{dx}$. **3**

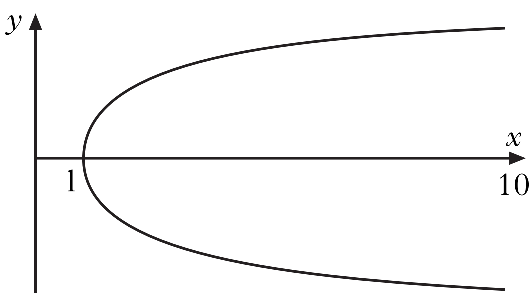
B3. Express $y = \dfrac{4x-3}{x(x^2+3)}$, $x \neq 0$, in partial fractions. **4**

B4. (*a*) Use integration by parts to show that $\int \ln x \, dx = x\ln x - x + c$. **2**

(*b*) A goblet consists of a bowl and a short stem. The diagram below shows the bowl section of the goblet (on its side).

The equation of the upper half of the curve is $y = 2\sqrt{\ln x}$ for $1 \leq x \leq 10$.





Given that the stem has length 1 and the overall height is 10, what is the capacity of the bowl? **4**

Marks

B5. (*a*) Use the standard formulas for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n}(6r^2 - r) = \tfrac{1}{2}n(n+1)(4n+1).$$ **3**

(*b*) Hence evaluate $\sum_{r=5}^{10}(6r^2 - r)$. **2**

B6. Newton's law of cooling states that a body loses heat at a rate which is proportional to the difference in temperature between itself and its surroundings. So, in a room with constant temperature 22 °C, the temperature T °C of a body after a time t minutes satisfies

$$\frac{dT}{dt} = k(T - 22)$$

where k is a negative constant.

(*a*) Hence show that T can be expressed in the form $T = Ae^{kt} + 22$ for some arbitrary constant A. **4**

(*b*) In a restaurant, where the temperature remains constant at 22 °C, a freshly baked roll, with temperature 82 °C, is placed on a cooling tray. After 5 minutes, the temperature of the roll has fallen by 20 degrees.

Calculate the values of A and k.

Write down an expression for the temperature of the roll after t minutes.

Supposing the roll remains uneaten after a further 5 minutes, what will its temperature be? **5**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]

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