X204/701

NATIONAL
QUALIFICATIONS
2007

TUESDAY, 22 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Mechanics

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**

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Section A (Mechanics 1 and 2)

Marks

Answer all the questions.

Candidates should observe that g m s^{-2} denotes the magnitude of the acceleration due to gravity.

Where appropriate, take its magnitude to be 9·8 m s^{-2}.

A1. The position of a remote controlled model boat on a pond, relative to a rectangular coordinate system with origin O, is given by

$$\mathbf{r} = (3t^2 - 12t + 5)\mathbf{i} + (4t - t^2)\mathbf{j},$$

where $\mathbf{i}$, $\mathbf{j}$ are unit vectors in the Ox and Oy directions respectively, t is time measured in seconds and distances are measured in metres.

Calculate the distance of the boat from the origin O when it comes to instantaneous rest. **4**

A2. A piston in a machine moves with simple harmonic motion about a point O. The period of the motion is 4π seconds. When the piston is 0·2 m from O its speed is 0·8 m s^{-1}.

Calculate the amplitude of the motion. **4**

A3. Two small spheres A and B have masses $3m$ kg and $2m$ kg respectively. They are moving towards each other, from opposite directions, in a straight line on a smooth horizontal surface. Each sphere has speed U m s^{-1} when they collide head on. After the collision, the direction of motion of B is reversed but its speed is unchanged.

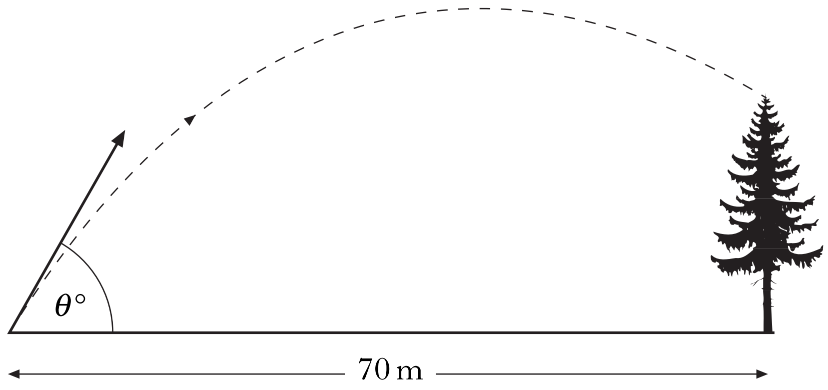
Show that the direction of motion of A is also reversed and find its speed in terms of U. **3**

A4. An athlete runs a 200 metre race, along a straight horizontal track, in 30 seconds. She accelerates uniformly from rest for 4 seconds, reaching a maximum speed of V m s^{-1}. She runs at this speed for 24 seconds before decelerating uniformly for the final 2 seconds, finishing the race with speed $(V - 6)$ m s^{-1}.

Sketch the speed-time graph for the race and calculate the value of V. **4**

Marks

A5. A golfer strikes a golf ball on a horizontal range, projecting the ball with speed 30 $\mathrm{m\,s^{-1}}$ at an angle $\theta°$ to the horizontal. After 3 seconds, the ball hits the top of a tree, which is situated at a horizontal distance of 70 metres from the point of projection.



Calculate the height of the tree. **4**

A6. A car of mass 1400 kg travels with constant speed, V $\mathrm{m\,s^{-1}}$, along a straight, horizontal stretch of road. A constant resistive force of magnitude 500 newtons acts on the car. The engine of the car is working at a rate of 15 kW. Calculate the value of V. **2**

When the car travels up a slight incline of $\theta°$ to the horizontal, its speed, which is again constant, is $\frac{1}{2}V$ $\mathrm{m\,s^{-1}}$. Assuming that the resistive force remains the same as before and that the engine continues to work at a rate of 15 kW, calculate the value of θ. **4**

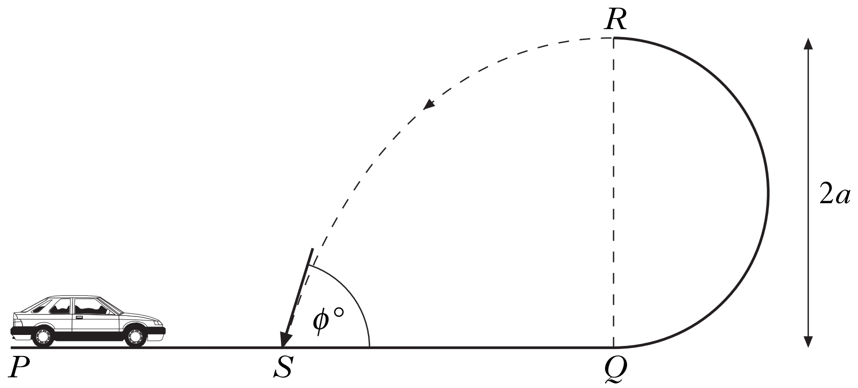
A7. At a building site, bricks of mass 2 kg slide down a straight chute into a container. The chute is rough and inclined at 30° to the horizontal. The distance travelled down the chute by each brick is 10 metres. A brick is released from rest at the top of the chute. When the brick reaches the bottom of the chute its speed is 4 $\mathrm{m\,s^{-1}}$.

Calculate the coefficient of friction between the brick and the chute. **6**

A8. A car and a motorcycle are travelling along straight, horizontal roads which intersect at right angles at a point O. The car is travelling northwards at a constant speed, while the motorcycle is travelling eastwards at twice the speed of the car. At the instant when the motorcycle passes through O, the car is 40 metres south of O. Calculate the minimum distance between the car and the motorcycle. **7**

[Turn over

A9. The diagram below shows a smooth plastic track. The section PQ is horizontal and the section QR is semi-circular and in the same plane as PQ. The diameter QR is vertical and has length $2a$ metres.



A toy car is projected along PQ with speed $3\sqrt{ga}$ m s^{-1}. The car travels around the track to R, where it leaves the track horizontally, landing on PQ at the point S, where the angle between the car's trajectory and the line SQ is $\phi°$.

(*a*) Find the speed of the car at R, expressing your answer in the form $\sqrt{kga}$, where k is a constant. **3**

(*b*) Show that at R the car is in contact with the track. **2**

(*c*) Show that $SQ = 2\sqrt{5}a$ metres. **3**

(*d*) Find the exact value of $\tan\phi°$. **2**

A10. A rubber ball of mass m kg falls vertically into a tank of water. When the ball is x metres below the surface of the water and moving downwards with speed v m s^{-1}, the water exerts a resistive force of magnitude $2mv^2$ newtons and an upward buoyancy force of magnitude three times the weight of the ball.

(*a*) Show that the downward motion of the ball can be modelled by the differential equation

$$v\frac{dv}{dx} = -2(v^2 + g).$$ **2**

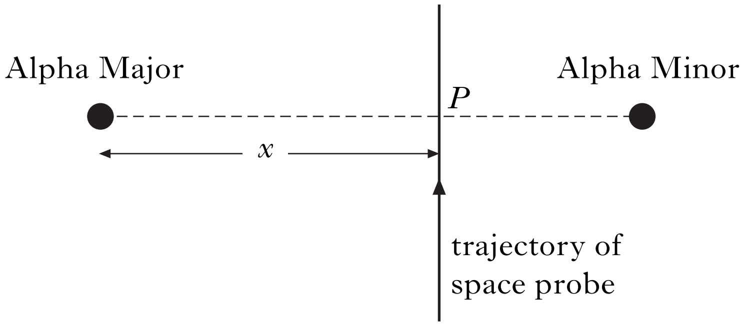
(*b*) The ball enters the water with speed U m s^{-1}. By solving the equation in (*a*), show that

$$v^2 + g = (U^2 + g)e^{-4x}.$$ **5**

(*c*) In the case when $U = 4{\cdot}9$, calculate, to the nearest centimetre, the greatest depth below the surface of the water reached by the ball. **3**

Marks

A11. (*a*) The trajectory of a space probe passes between two planets Alpha Major and Alpha Minor which are L metres apart and with masses $4M$ kg and M kg respectively. The trajectory of the space probe intersects the line joining the centres of the planets at right angles at P, which is a distance x metres from Alpha Major.



The only forces acting on the space probe are the gravitational forces from the two planets. These forces obey the inverse square law. Assuming that these gravitational forces are equal and opposite, show that at P

$$3x^2 - 8xL + 4L^2 = 0.$$

Hence find the value of x, expressing your answer in terms of L. **5**

(*b*) The space probe is then manoeuvred into a circular orbit about Alpha Major at a height $2R$ metres above the surface of the planet, where R metres is the radius of Alpha Major.

Show that the period of the orbit is

$$T = \pi\sqrt{\frac{27R^3}{GM}},$$

where G is the universal constant of gravitation in its usual units. **5**

[*END OF SECTION A*]

[Turn over for Section B on *Page six*

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

B1. Find the exact value of $\int_0^{\pi/6} x \sin 3x \, dx$. **5**

B2. Use the binomial theorem to expand $\left(x^3 - \frac{2}{x}\right)^4$ and simplify your answer. **4**

B3. A curve is defined parametrically by $x = \frac{t}{t^2+1}$, $y = \frac{t-1}{t^2+1}$.

Obtain $\frac{dy}{dx}$ as a function of t. **5**

B4. For the matrix $A = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix}$, find the values of λ such that the matrix is singular. **3**

Write down the matrix A^{-1} when $\lambda = 3$. **1**

B5. Obtain the solution of the differential equation

$$x\frac{dy}{dx} - y = x^2 e^x$$

for which $y = 2$ when $x = 1$. **5**

B6. Express $\frac{8}{x(x+2)(x+4)}$ in partial fractions. **4**

Calculate the area under the curve

$$y = \frac{8}{x^3 + 6x^2 + 8x}$$

between $x = 1$ and $x = 2$. Express your answer in the form $\ln \frac{a}{b}$, where a and b are positive integers. **5**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]

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